Web Appendix for "Estimation of Matrix Exponential Unbalanced Panel Data Model with Fixed Effects"

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This web appendix presents the proofs of the technical results, the details of the empirical exercise and the additional simulation results. More specifically, Section A includes some lemmas that are essential for our theoretical results. The proofs of some lemmas are given in Section B. Sections C and D provide the proofs of the main technical results. Section E provides some details on our empirical application. Section F presents the details on the identification conditions. Section G includes the pseudo estimation algorithms. Finally, Section H provides the additional simulation results.

A Some Useful Lemmas

The following lemmas are useful in the proofs of the theorems in the paper. Lemma A.1 can be found in Kelejian and Prucha (1999). Lemma A.3 can be found in Lin and Lee (2010), Lemma A.4 can be found in Debarsy et al. (2015), Lemma A.6 can be found in Lee (2007a), Lemma A.7 can be found in Lin and Lee (2010). The proofs of Lemma A.2 and A.5 can be found in section B.

Lemma A.1. *Let $\{A_N\}$ and $\{B_N\}$ be two sequences of $N \times N$ matrices that are uniformly bounded in both row sum and column sum matrix norms. Let $\{C_N\}$ be a sequence of conformable matrices whose elements are uniformly $O(h_n^{-1})$. Then,*

(i) the sequence $\{A_N B_N\}$ are uniformly bounded in both row sum and column sum matrix norms,

(ii) the elements of $\{A_N\}$ are uniformly bounded and $\operatorname{tr}(A_N) = O(N)$, and

(iii) the elements of $\{A_N C_N\}$ and $\{C_N A_N\}$ are uniformly $O(h_n^{-1})$.

Lemma A.2. *Under our assumptions in the paper, we have*

(i) $\mathbb{Q}_{\mathbb{C}}(\tau)$ is uniformly bounded in both row sum and column sum matrix norms, uniformly in $\tau \in \Delta_\tau$,

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(ii) $\mathbb{Q}_{\mathbb{X}}(\tau)$ *is uniformly bounded in both row sum and column sum matrix norms, uniformly in* $\tau \in \Delta_\tau$*,*

(iii) The elements of $\mathbb{P}_{\mathbb{C}}(\tau)$ *has the uniform order* $O(\max\{1/n, 1/T\})$*, uniformly in* $\tau \in \Delta_\tau$*.*

(iv) Let $\{A_N\}$ *be a sequence of* $N \times N$ *matrices that are uniformly bounded in both row sum and column sum matrix norms and* $\{C_N\}$ *be a sequence of* $N \times N$ *matrices whose elements are uniformly* $O(h_n^{-1})$*.*

(a) $\frac{1}{N}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)A_N\right) = \frac{1}{N}\mathrm{tr}\left(A_N\right) + O\left(\max\{1/n, 1/T\}\right)$*, uniformly in* $\tau \in \Delta_\tau$*.*

(b) $\frac{1}{N}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)C_N\right) = \frac{1}{N}\mathrm{tr}\left(C_N\right) + O\left(1/\max\{T, h_n\}\right) = O(h_n^{-1}) + O\left(1/\max\{T, h_n\}\right)$*, uniformly in* $\tau \in \Delta_\tau$*.*

Remark 1. *A similar result in Lee (2004, Lemma A.9) shows that* $\frac{1}{n}\mathrm{tr}(M_nA_n) = \frac{1}{n}\mathrm{tr}(A_n) + o(1)$*, where* A_n *are uniformly bounded in both row and column sums and* $M_n = I_n - X_n(X_n'X_n)^{-1}X_n'$*. This result follows from* $\frac{1}{n}\mathrm{tr}(M_nA_n) = \frac{1}{n}\mathrm{tr}(A_n) - \frac{1}{n}\mathrm{tr}\left((X_n'X_n)^{-1}X_n'A_nX_n\right)$*. Under the assumption that* $\lim_{n\to\infty}\frac{1}{n}X_n'X_n$ *exists and is non-singular, Lee (2004) shows that the elements of* $k\times k$ *matrices* $(\frac{1}{n}X_n'X_n)^{-1}$ *and* $\frac{1}{n}X_n'A_nX_n$ *are uniformly bounded, where* k *is the number of columns in* X_n*. Thus, it follows that* $\mathrm{tr}\left((X_n'X_n)^{-1}X_n'A_nX_n\right) = O(1)$*, which implies that* $\frac{1}{n}\mathrm{tr}(M_nA_n) = \frac{1}{n}\mathrm{tr}(A_n) + o(1)$*. This result cannot be used in our case for the terms such as* $\frac{1}{N}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)A_N\right)$ *and* $\frac{1}{N}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)C_N\right)$ *because* $\mathbb{C}(\tau)$ *is an* $N \times (n+T-1)$ *matrix, i.e., its column dimension depends on* n *and* T*.*

Lemma A.3. *Let* $\{A_N\}$ *be a sequence of* $N \times N$ *matrices such that either* $\|A_N\|_\infty$ *or* $\|A_N\|_1$ *is bounded. Suppose that the elements of* A_N *are* $O\left(h_n^{-1}\right)$ *uniformly. Assume that the elements of the innovation vector* ϵ *have zero mean and finite variance, and are mutually independent. Let* c_N *be an* $N \times 1$ *vector with elements of uniform order* $O(h_n^{-1/2})$*. Then,*

(i) $\mathrm{E}\left(\epsilon' A_N \epsilon\right) = O\left(\frac{N}{h_n}\right)$*,* *(ii)* $\mathrm{Var}\left(\epsilon' A_N \epsilon\right) = O\left(\frac{N}{h_n}\right)$*,*

(iii) $\epsilon' A_N \epsilon = O_p\left(\frac{N}{h_n}\right)$*,* *(iv)* $\epsilon' A_N \epsilon - \mathrm{E}\left(\epsilon' A_N \epsilon\right) = O_p\left(\left(\frac{N}{h_n}\right)^{\frac{1}{2}}\right)$*,*

(v) $c_N' A_N \epsilon = O_p\left(\left(\frac{N}{h_n}\right)^{\frac{1}{2}}\right)$*, if* $\|A_N\|_1$ *is bounded.*

Lemma A.4. *Let* A_n *be any* $n \times n$ *matrix that is uniformly bounded in row sum and column sum matrix norms and* $a_n = o_p(1)$*. Then* $\left\|e^{a_nA_n} - I_n\right\|_\infty = o_p(1)$ *and* $\left\|e^{a_nA_n} - I_n\right\|_1 = o_p(1)$*.*

Lemma A.5. *Suppose that* $\{A_N\}$ *and* $\{B_N\}$ *are two sequences of* $N\times N$ *matrices that are uniformly bounded in either row sum or column sum matrix norms. Under our assumptions in the paper,* $\mathrm{tr}(A_N\mathbb{P}_X(\tau)B_N) = O(1)$*, uniformly in* $\tau \in \Delta_\tau$*.*

Lemma A.6. *Assume that the elements* $\{\epsilon_i\}$ *in the innovation vector* ϵ *are independent and identically distributed with mean zero and finite variance* σ^2*. Let* $E(\epsilon_i^3) = \eta_3$ *and* $E(\epsilon_i^4) = \eta_4$*. For any* $N \times N$ *matrices* A_N *and* B_N *of constants, define* $A_N^s = A_N + A_N'$ *and* $B_N^s = B_N + B_N'$*. Then,* *(i)* $\mathrm{E}(\epsilon' A_N\epsilon \times \epsilon' B_N\epsilon) = (\eta_4 - 3\sigma^4)vec_D'(A_N)vec_D(B_N) + \sigma^4\left(\mathrm{tr}(A_N)\mathrm{tr}(B_N) + \mathrm{tr}(A_N B_N^s)\right)$,

(ii) $\mathrm{E}(A_N\epsilon \times \epsilon' B_N\epsilon) = A_N vec_D(B_N)\eta_3$,

(iii) $\mathrm{E}(\epsilon' B_N\epsilon \times \epsilon' A_N) = \eta_3 vec_D'(B_N)A_N$.

Lemma A.7. *Assume that the elements $\{\epsilon_i\}$ in the innovation vector ϵ are independent and distributed with mean zero and finite variance σ_i^2. Let $E(\epsilon_i^3) = \eta_{i3}$ and $E(\epsilon_i^4) = \eta_{i4}$. For any $N \times N$ matrix $A_N = [a_{ij}]$ and $B_N = [b_{ij}]$ of constants, define $A_N^s = A_N + A_N'$ and $B_N^s = B_N + B_N'$. Let c_N be an $N \times 1$ vector of elements c_i and $\Gamma = \mathrm{diag}\left(\sigma_1^2, \dots, \sigma_N^2\right)$. Then,*

(i) $\mathrm{E}(\epsilon' A_N\epsilon \times \epsilon' B_N\epsilon) = \sum_{i=1}^N a_{ii}b_{ii}\left(\eta_{i4} - 3\sigma_i^4\right) + \mathrm{tr}(\Gamma A_N)\mathrm{tr}(\Gamma B_N) + \mathrm{tr}(\Gamma A_N \Gamma B_N^s)$,

(ii) $\mathrm{E}(\epsilon' A_N\epsilon \times c_N'\epsilon) = \sum_{i=1}^N a_{ii}c_i\eta_{i3}$,

(iii) $\mathrm{E}(A_N\epsilon \times \epsilon' B_N\epsilon) = A_N\sum_{i=1}^N b_{ii}\eta_{i3}$,

(iv) $\mathrm{E}(\epsilon' A_N\epsilon) = \mathrm{tr}(\Gamma A_N) = \sum_{i=1}^N a_{ii}\sigma_i^2$.

B Proofs of Lemmas

Proof of Lemma A.2.

Proof of (i)**.** The order analysis of these terms becomes tractable, when the identification restriction is imposed as $\lambda_1 = 0$. This is an equivalent way of achieving the identification restriction. Then, we can write $C_\lambda^* = [0_{n_1\times(T-1)}; \mathrm{blkdiag}(l_{n_2}, \dots, l_{n_T})]$, where the semicolon sign ; means vertical stack. Let $\mathbb{C}_\mu(\tau) = \mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}C_\mu$, $\mathbb{C}_\lambda(\tau) = \mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}C_\lambda^*$, $\overline{\mathbb{C}}_{11}(\tau) = \mathbb{C}_\mu'\mathbb{C}_\mu$, $\overline{\mathbb{C}}_{12}(\tau) = \mathbb{C}_\mu'\mathbb{C}_\lambda$, $\overline{\mathbb{C}}_{21}(\tau) = \mathbb{C}_\lambda'\mathbb{C}_\mu$, $\overline{\mathbb{C}}_{22}(\tau) = \mathbb{C}_\lambda'\mathbb{C}_\lambda$ and $\mathbb{B}(\tau) = \mathbb{C}_\mu'(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)$. Recall that $\mathbb{C}(\tau) = \mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}C = [\mathbb{C}_\mu(\tau) \quad \mathbb{C}_\lambda(\tau)]$. By the formula for the inverse of a partitioned matrix,

$$
\begin{aligned}
[\mathbb{C}'(\tau)\mathbb{C}(\tau)]^{-1} &= \begin{bmatrix} \overline{\mathbb{C}}_{11}(\tau) & \overline{\mathbb{C}}_{12}(\tau) \\ \overline{\mathbb{C}}_{21}(\tau) & \overline{\mathbb{C}}_{22}(\tau) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \mathbb{B}^{-1}(\tau) & -\mathbb{B}^{-1}(\tau)\overline{\mathbb{C}}_{12}(\tau)\overline{\mathbb{C}}_{22}^{-1}(\tau) \\ -\overline{\mathbb{C}}_{22}^{-1}(\tau)\overline{\mathbb{C}}_{12}(\tau)\mathbb{B}^{-1}(\tau) & \overline{\mathbb{C}}_{22}^{-1}(\tau) + \overline{\mathbb{C}}_{22}^{-1}(\tau)\overline{\mathbb{C}}_{12}'(\tau)\mathbb{B}^{-1}(\tau)\overline{\mathbb{C}}_{12}(\tau)\overline{\mathbb{C}}_{22}^{-1}(\tau) \end{bmatrix}.
\end{aligned}
$$

Substituting this expression into the definition of $\mathbb{Q}_{\mathbb{C}}(\tau)$, after a little bit of tedious derivation we obtain $\mathbb{Q}_{\mathbb{C}}(\tau) = \mathbb{Q}_{\mathbb{C}_\lambda}(\tau) - \mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)[\mathbb{C}_\mu'(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)]^{-1}\mathbb{C}_\mu'(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)$.

Since $\mathbb{C}_\lambda(\tau) = [0_{n_1\times(T-1)}; \mathrm{blkdiag}(e^{\tau M_2}l_{n_2}, \dots, e^{\tau M_T}l_{n_T})]$, for the first element on the right hand side of the above equation, we have $\mathbb{Q}_{\mathbb{C}_\lambda}(\tau) = \mathrm{blkdiag}(K_1(\tau), \dots, K_T(\tau))$, where $K_1(\tau) = I_{n_1}$ and $K_t(\tau) = I_{n_t} - \frac{1}{n_t}e^{\tau M_t}l_{n_t}[\frac{1}{n_t}l_{n_t}'e^{\tau M_t'}e^{\tau M_t}l_{n_t}]^{-1}l_{n_t}'e^{\tau M_t'}$ for $t = 2, \dots, T$. By Assumptions 3 and 4, $e^{\tau M_t}$ is bounded in row sum matrix norm uniformly in $\tau \in \Delta_\tau$. Hence, the elements of $e^{\tau M_t}l_{n_t}$ is bounded uniformly in $\tau \in \Delta_\tau$. Therefore, $\frac{1}{n_t}l_{n_t}'e^{\tau M_t'}e^{\tau M_t}l_{n_t}$ is bounded uniformly in $\tau \in \Delta_\tau$. It is also bounded away from zero uniformly in $\tau \in \Delta_\tau$, because it is a sum of squares. Then, the elements of $\frac{1}{n_t}e^{\tau M_t}l_{n_t}[\frac{1}{n_t}l_{n_t}'e^{\tau M_t'}e^{\tau M_t}l_{n_t}]^{-1}l_{n_t}'e^{\tau M_t'}$ has the uniform order $O(1/n_t)$, which is equal to

$O(1/n)$ by Assumption 2. Then, $\frac{1}{n_t}e^{\tau M_t}l_{n_t}[\frac{1}{n_t}l_{n_t}^{'}e^{\tau M_t^{\prime}}e^{\tau M_t}l_{n_t}]^{-1}l_{n_t}^{'}e^{\tau M_t^{\prime}}$ is bounded in row sum and column sum matrix norms uniformly in $\tau \in \Delta_\tau$ for $t = 2, \ldots, T$. Therefore, $K_t(\tau)$ is uniformly bounded in both row and column sum matrix norms uniformly in $\tau \in \Delta_\tau$ for $t = 2, \ldots, T$. This implies that $\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)$ is uniformly bounded in both row and column sum matrix norms uniformly in $\tau \in \Delta_\tau$.

The second term on the right hand side of the above equation can be partitioned into $T \times T$ tiles. The (s,t)th tile can be written as $-\frac{1}{T}K_s(\tau)e^{\tau M_s}C_s[\frac{1}{T}\sum_{t=1}^{T}C_t^{\prime}e^{\tau M_t^{\prime}}K_t(\tau)e^{\tau M_t}C_t]^{-1}C_t^{\prime}e^{\tau M_t^{\prime}}K_t(\tau)$. By Assumption 5, $e^{\tau M_s}C_s[\frac{1}{T}\sum_{t=1}^{T}C_t^{\prime}e^{\tau M_t^{\prime}}K_t(\tau)e^{\tau M_t}C_t]^{-1}C_t^{\prime}e^{\tau M_t^{\prime}}$ is bounded in row sum and column sum matrix norms uniformly in $\tau \in \Delta_\tau$. Then, the elements of the (s,t)th block has the uniform order $O(1/T)$, uniformly in $\tau \in \Delta_\tau$. Hence, the second term on the right-hand side of the above equation is also bounded in both row and column sum matrix norms uniformly in $\tau \in \Delta_\tau$. Therefore, $\mathbb{Q}_{\mathbb{C}}(\tau)$ is bounded in both row sum and column sum matrix norms uniformly in $\tau \in \Delta_\tau$.

Proof of (ii)**.** Let $\overline{\mathbb{X}}(\tau) = [\frac{1}{N}\mathbb{X}^{'}(\tau)\mathbb{X}(\tau)]^{-1}$ and denote its (j,k)th element by $\overline{\mathbb{X}}_{jk}(\tau)$. By Assumption 7, there exists a constant $a_{\overline{X}}$ such that $\left|\overline{\mathbb{X}}_{jk}(\tau)\right| \leq a_{\overline{X}}$ uniformly in $\tau \in \Delta_\tau$ for large enough N. Also, by Assumption 7, the elements of X are non-stochastic and bounded. In the previous part, we showed $\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)$ is bounded in row sum and column sum matrix norms uniformly in $\tau \in \Delta_\tau$. Also, by Assumptions 3 and 4, $\mathbf{e^{\tau M}}$ is bounded in row sum and column sum matrix norms uniformly in $\tau \in \Delta_\tau$. Therefore, the elements of $\mathbb{X}(\tau) = \mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e^{\tau M}}X$ are bounded uniformly in $\tau \in \Delta_\tau$ by Lemma A.1.

Let $\mathbb{X}_{jk}(\tau)$ be the (j,k)th element of $\mathbb{X}(\tau)$. Then there exists a constant a_X such that $|\mathbb{X}_{jk}(\tau)| \leq a_X$ uniformly in $\tau \in \Delta_\tau$. Let $\mathbb{P}_{jl}(\tau)$ be the (j,l)th element of $\mathbb{P}_{\mathbb{X}}(\tau) = \frac{1}{N}\mathbb{X}(\tau)[\frac{1}{N}\mathbb{X}^{'}(\tau)\mathbb{X}(\tau)]^{-1}\mathbb{X}^{'}(\tau)$. Then $\sum_{j=1}^{N}|\mathbb{P}_{jl}(\tau)| \leq \frac{1}{N}\sum_{j=1}^{N}\sum_{r=1}^{k}\sum_{s=1}^{k}\left|\overline{\mathbb{X}}_{rs}(\tau)\mathbb{X}_{jr}(\tau)\mathbb{X}_{ls}(\tau)\right| \leq k^2 a_{\overline{X}}a_X^2$ uniformly in $\tau \in \Delta_\tau$ for all $l = 1, \ldots, N$. Also $\sum_{l=1}^{N}|\mathbb{P}_{jl}(\tau)| \leq \frac{1}{N}\sum_{l=1}^{N}\sum_{r=1}^{k}\sum_{s=1}^{k}\left|\overline{\mathbb{X}}_{rs}(\tau)\mathbb{X}_{jr}(\tau)\mathbb{X}_{ls}(\tau)\right| \leq k^2 a_{\overline{X}}a_X^2$ for all $j = 1, \ldots, N$. Then $\mathbb{P}_{\mathbb{X}}(\tau)$ is bounded in row sum and column sum matrix norms uniformly in $\tau \in \Delta_\tau$. Thus, $\mathbb{Q}_{\mathbb{X}}(\tau)$ is bounded in row sum and column sum matrix norms uniformly in $\tau \in \Delta_\tau$.

Proof of (iii)**.** Recall that in part (i), we showed $\mathbb{Q}_{\mathbb{C}}(\tau) = \mathbb{Q}_{\mathbb{C}_\lambda}(\tau) - \mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)[\mathbb{C}_\mu^{'}(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)]^{-1}\mathbb{C}_\mu^{'}(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)$. Hence, $\mathbb{P}_{\mathbb{C}}(\tau) = I_N - \mathbb{Q}_{\mathbb{C}}(\tau) = I_N - \mathbb{Q}_{\mathbb{C}_\lambda}(\tau) + \mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)[\mathbb{C}_\mu^{'}(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)]^{-1}\mathbb{C}_\mu^{'}(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau) = \mathbb{P}_{\mathbb{C}_\lambda}(\tau) + \mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)[\mathbb{C}_\mu^{'}(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)]^{-1}\mathbb{C}_\mu^{'}(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)$.

In part (i), we showed that $\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)[\mathbb{C}_\mu^{'}(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)\mathbb{C}_\mu(\tau)]^{-1}\mathbb{C}_\mu^{'}(\tau)\mathbb{Q}_{\mathbb{C}_\lambda}(\tau)$ can be partitioned to $T \times T$ tiles, and the (s,t)th tile can be written as $-\frac{1}{T}K_s(\tau)e^{\tau M_s}C_s[\frac{1}{T}\sum_{t=1}^{T}C_t^{\prime}e^{\tau M_t^{\prime}}K_t(\tau)e^{\tau M_t}C_t]^{-1}C_t^{\prime}e^{\tau M_t^{\prime}}K_t(\tau)$. We also concluded that the elements of the (s,t)th block (and therefore the entire term) has the uniform order $O(1/T)$, uniformly in $\tau \in \Delta_\tau$.

Also, in part (i), we showed that $\mathbb{Q}_{\mathbb{C}_\lambda}(\tau) = \text{blkdiag}(K_1(\tau), \ldots, K_T(\tau))$, where $K_1(\tau) = I_{n_1}$ and $K_t(\tau) = I_{n_t} - \frac{1}{n_t}e^{\tau M_t}l_{n_t}[\frac{1}{n_t}l_{n_t}^{'}e^{\tau M_t^{\prime}}e^{\tau M_t}l_{n_t}]^{-1}l_{n_t}^{'}e^{\tau M_t^{\prime}}$ for $t = 2, \ldots, T$. Our analysis indicated that the elements of $\frac{1}{n_t}e^{\tau M_t}l_{n_t}[\frac{1}{n_t}l_{n_t}^{'}e^{\tau M_t^{\prime}}e^{\tau M_t}l_{n_t}]^{-1}l_{n_t}^{'}e^{\tau M_t^{\prime}}$ has the uniform order $O(1/n)$. Therefore, the elements of $\mathbb{P}_{\mathbb{C}_\lambda}(\tau)$ has the uniform order $O(1/n)$, uniformly in $\tau \in \Delta_\tau$.

Combining these two terms, the elements of $\mathbb{P}_{\mathbb{C}}(\tau)$ has the uniform order $O\left(\max\{1/n,\, 1/T\}\right)$, uniformly in $\tau \in \Delta_\tau$.

Proof of *(iv)***.** (a) By definition, $\mathbb{Q}_{\mathbb{C}}(\tau) = I_N - \mathbb{P}_{\mathbb{C}}(\tau)$. Then, it follows that $\frac{1}{N}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)A_N\right) = \frac{1}{N}\mathrm{tr}\left(A_N\right) - \frac{1}{N}\mathrm{tr}\left(\mathbb{P}_{\mathbb{C}}(\tau)A_N\right)$. Note that the elements of $\mathbb{P}_{\mathbb{C}}(\tau)$ has the uniform order $O\left(\max\left\{1/n, 1/T\right\}\right)$ by part (iii). Then, by Lemma A.1 (iii), the elements of $\mathbb{P}_{\mathbb{C}}(\tau)A_N$ are uniformly $O\left(\max\left\{1/n, 1/T\right\}\right)$. Thus, $\frac{1}{N}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)A_N\right) = \frac{1}{N}\mathrm{tr}\left(A_N\right) + O\left(\max\left\{1/n, 1/T\right\}\right)$, uniformly in $\tau \in \Delta_\tau$. (b) Similarly, we have $\frac{1}{N}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)C_N\right) = \frac{1}{N}\mathrm{tr}\left(C_N\right) - \frac{1}{N}\mathrm{tr}\left(\mathbb{P}_{\mathbb{C}}(\tau)C_N\right)$. Since the elements of C_N are uniformly $O(h_n^{-1})$, we have $\frac{1}{N}\mathrm{tr}\left(C_N\right) = O(h_n^{-1})$ by Lemma A.1 (ii). Since the elements of $\mathbb{P}_{\mathbb{C}}(\tau)$ are uniformly $O\left(\max\left\{1/n, 1/T\right\}\right)$ and that of C_N are uniformly $O(h_n^{-1})$, the order of $\frac{1}{N}\mathrm{tr}\left(\mathbb{P}_{\mathbb{C}}(\tau)C_N\right)$ is either uniformly $O(1/T)$ or $O(1/h_n)$. Therefore, $\frac{1}{N}\mathrm{tr}\left(\mathbb{P}_{\mathbb{C}}(\tau)C_N\right) = O\left(1/\max\{T, h_n\}\right)$ uniformly. Then, we have $\frac{1}{N}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)C_N\right) = \frac{1}{N}\mathrm{tr}\left(C_N\right) + O\left(1/\max\{T, h_n\}\right) = O(h_n^{-1}) + O\left(1/\max\{T, h_n\}\right)$, uniformly in $\tau \in \Delta_\tau$.

Proof of Lemma A.5.

From the proof of Lemma A.2, we know that the elements of $\mathbb{X}(\tau)$ and $\bar{\mathbb{X}}(\tau) = [\frac{1}{N}\mathbb{X}'(\tau)\mathbb{X}(\tau)]^{-1}$ are uniformly bounded in $\tau \in \Delta_\tau$. By the assumption of the lemma, A_N and B_N are uniformly bounded in row or column sum norm. Then by Lemma A.1, $B_N A_N$ is also uniformly bounded in row or column sum norm. By Lemma A.6 of Lee (2004), the elements of $\frac{1}{N}\mathbb{X}'(\tau)B_N A_N \mathbb{X}(\tau)$ are uniformly bounded. Then $\mathrm{tr}(A_N \mathbb{P}_X(\tau) B_N) = \mathrm{tr}[A_N \mathbb{X}(\tau)(\mathbb{X}'(\tau)\mathbb{X}(\tau))^{-1}\mathbb{X}'(\tau)B_N] = \mathrm{tr}[(\frac{1}{N}\mathbb{X}'(\tau)\mathbb{X}(\tau))^{-1}(\frac{1}{N}\mathbb{X}'(\tau)B_N A_N \mathbb{X}(\tau))] = O(1)$ uniformly in $\tau \in \Delta_\tau$ since there are fixed number of k independent variables.

C Proofs of theorems

C.1 Proof of Theorem 3.1

Given Assumption 6, we need to prove $\sup_{\zeta \in \Delta} \frac{1}{N_1}\left\| S^{*c}(\zeta) - \bar{S}^{*c}(\zeta)\right\| \xrightarrow{p} 0$. Note that we can express $S^{*c}(\zeta) - \bar{S}^{*c}(\zeta)$ as

$$S^{*c}(\zeta) - \bar{S}^{*c}(\zeta) = \begin{cases} \alpha: & \frac{\widehat{\sigma}_\epsilon^{*2}(\zeta) - \bar{\sigma}_\epsilon^{*2}(\zeta)}{\bar{\sigma}_\epsilon^{*2}(\zeta)\widehat{\sigma}_\epsilon^{*2}(\zeta)} y'\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\widehat{\epsilon}(\zeta) - \frac{1}{\bar{\sigma}_\epsilon^{*2}(\zeta)}\Big(y'\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\widehat{\epsilon}(\zeta) \\ & \quad -\mathrm{E}\left(y'\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\bar{\epsilon}(\zeta)\right)\Big), \\ \tau: & \frac{\widehat{\sigma}_\epsilon^{*2}(\zeta) - \bar{\sigma}_\epsilon^{*2}(\zeta)}{\bar{\sigma}_\epsilon^{*2}(\zeta)\widehat{\sigma}_\epsilon^{*2}(\zeta)}\widehat{\epsilon}'(\zeta)\mathbf{M}\widehat{\epsilon}(\zeta) - \frac{1}{\bar{\sigma}_\epsilon^{*2}(\zeta)}\left(\widehat{\epsilon}'(\zeta)\mathbf{M}\widehat{\epsilon}(\zeta) - \mathrm{E}(\bar{\epsilon}'(\zeta)\mathbf{M}\bar{\epsilon}(\zeta))\right). \end{cases}$$

So we need to prove the following results:

(i) $\inf_{\zeta \in \Delta} \bar{\sigma}_\epsilon^{*2}(\zeta) > c > 0$ for some positive number c,

(ii) $\sup_{\zeta \in \Delta}\left|\widehat{\sigma}_\epsilon^{*2}(\zeta) - \bar{\sigma}_\epsilon^{*2}(\zeta)\right| = o_p(1)$,

(iii) $\sup_{\zeta \in \Delta} \frac{1}{N_1}\left\| y'\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\widehat{\epsilon}(\zeta) - \mathrm{E}\left(y'\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\bar{\epsilon}(\zeta)\right)\right\| = o_p(1)$,

(iv) $\sup_{\zeta\in\Delta}\frac{1}{N_1}\left\|\widehat{\epsilon}^{'}(\zeta)\mathbf{M}\widehat{\epsilon}(\zeta)-\mathrm{E}\left(\bar{\epsilon}^{'}(\zeta)\mathbf{M}\bar{\epsilon}(\zeta)\right)\right\|=o_p(1)$.

Proof of (i). Utilizing $\bar{\epsilon}(\zeta)=\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(y-\mathrm{E}(y)\right)+\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y$ in equation (3.13), we can express $\bar{\sigma}_{\epsilon}^{*2}(\zeta)$ as:

$$\begin{aligned}\bar{\sigma}_{\epsilon}^{*2}(\zeta)&=\frac{1}{N_1}\mathrm{E}\left(\bar{\epsilon}^{'}(\zeta)\bar{\epsilon}(\zeta)\right)=\frac{1}{N_1}\mathrm{E}\left((y-\mathrm{E}(y))^{'}\mathbf{P}(\zeta)(y-\mathrm{E}(y))+y^{'}\mathbf{Q}(\zeta)y\right)\\&=\frac{\sigma_{\epsilon0}^2}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbf{P}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)+\frac{1}{N_1}\mathrm{E}\left(y^{'}\mathbf{Q}(\zeta)y\right),\end{aligned}$$

where $\mathbf{Q}(\zeta)=\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)$ and $\mathbf{P}(\zeta)=\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)$. Using the fact that $y=\mathrm{E}(y)+\mathbf{G}^{-1}(\zeta_0)\epsilon$, we can further express $\bar{\sigma}_{\epsilon}^{*2}(\zeta)$ as

$$\bar{\sigma}_{\epsilon}^{*2}(\zeta)=\frac{\sigma_{\epsilon0}^2}{N_1}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{F}(\zeta)\right)+\frac{1}{N_1}\mathrm{E}(y)^{'}\mathbf{Q}(\zeta)\mathrm{E}(y), \tag{C.1}$$

where $\mathbf{F}(\zeta)=\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)$. For the first term in (C.1), we have

$$\begin{aligned}&\frac{\sigma_{\epsilon0}^2}{N_1}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{F}(\zeta)\right)\geq\frac{\sigma_{\epsilon0}^2}{N_1}\gamma_{\min}\left(\mathbf{F}(\zeta)\right)\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\right)\\&\geq\sigma_{\epsilon0}^2\gamma_{\min}(\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}})\gamma_{\min}(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}})\gamma_{\max}(\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}})^{-1}\gamma_{\max}(\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}^{'}}\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}})^{-1}>0,\end{aligned}$$

because the matrix exponential terms are positive definite. The second term in (C.1) is non-negative uniformly in $\zeta\in\Delta$ since $\mathbb{Q}_{\mathbb{X}}(\tau)$ is positive semi-definite.

Proof of (ii). We can express $\widehat{\epsilon}(\zeta)$ as $\widehat{\epsilon}(\zeta)=\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\tau)y-\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}X\widehat{\beta}^{*}(\tau)$, where $\widehat{\beta}^{*}(\zeta)=\left(\mathbb{X}^{'}(\tau)\mathbb{X}(\tau)\right)^{-1}\mathbb{X}^{'}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y$. Thus, we can further express $\widehat{\epsilon}(\zeta)$ as $\widehat{\epsilon}(\zeta)=\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\tau)y-\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\tau)y=\left(\mathbb{Q}_{\mathbb{X}}(\tau)+\mathbb{P}_{\mathbb{X}}(\tau)\right)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\tau)y-\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\tau)y=\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y$. Then, we have $\widehat{\sigma}_{\epsilon}^{*2}(\zeta)=\frac{1}{N_1}\widehat{\epsilon}^{'}(\zeta)\widehat{\epsilon}(\zeta)=\frac{1}{N_1}y^{'}\mathbf{Q}(\zeta)y$. Also in the proof of (i), we showed that $\bar{\sigma}_{\epsilon}^{*2}(\zeta)=\frac{\sigma_{\epsilon0}^2}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbf{P}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)+\frac{1}{N_1}\mathrm{E}\left(y^{'}\mathbf{Q}(\zeta)y\right)$. Thus,

$$\widehat{\sigma}_{\epsilon}^{*2}(\zeta)-\bar{\sigma}_{\epsilon}^{*2}(\zeta)=-\frac{\sigma_{\epsilon0}^2}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbf{P}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)+\frac{1}{N_1}\left(y^{'}\mathbf{Q}(\zeta)y-\mathrm{E}(y^{'}\mathbf{Q}(\zeta)y)\right). \tag{C.2}$$

For the first term on the right hand side of (C.2), we have

$$\begin{aligned}\frac{\sigma_{\epsilon0}^2}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbf{P}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)&=\frac{\sigma_{\epsilon0}^2}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)\\&\leq\frac{\sigma_{\epsilon0}^2}{N_1}\gamma_{\max}\left(\mathbf{F}(\zeta)\right)\gamma_{\max}^2\left(\mathbb{Q}_{\mathbb{C}}(\tau)\right)\mathrm{tr}\left(\mathbb{P}_{\mathbb{X}}(\tau)\right)=o(1),\end{aligned}$$

because $\gamma_{\max}\left(\mathbf{F}(\zeta)\right)\leq\gamma_{\max}(\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}})\gamma_{\max}(\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}^{'}}\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}})\gamma_{\max}(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}})\gamma_{\max}(\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}})<\infty$ by the fact that the norm of matrix exponential terms are finite, $\gamma_{\max}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\right)=1$ and $\mathrm{tr}\left(\mathbb{P}_{\mathbb{X}}(\tau)\right)=k$. Thus, we have $\sup_{\zeta\in\Delta}\left|\frac{\sigma_{\epsilon0}^2}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbf{P}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)\right|=o(1)$. For the second term on the right hand side of (C.2), we need to prove $\sup_{\zeta\in\Delta}\left|\frac{1}{N_1}\left(y^{'}\mathbf{Q}(\zeta)y-\mathrm{E}(y^{'}\mathbf{Q}(\zeta)y)\right)\right|=o_p(1)$, which follows from the point-wise convergence of $\frac{1}{N_1}\left(y^{'}\mathbf{Q}(\zeta)y-\mathrm{E}(y^{'}\mathbf{Q}(\zeta)y)\right)$ in each $\zeta\in\Delta$ and stochastic equicontinuity of $\frac{1}{N_1}y^{'}\mathbf{Q}(\zeta)y$. To prove the point-wise convergence, we have

$$\begin{aligned}&\frac{1}{N_1}\left(y^{'}\mathbf{Q}(\zeta)y-\mathrm{E}(y^{'}\mathbf{Q}(\zeta)y)\right)\\&=\frac{2}{N_1}\beta_0^{'}X^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{Q}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon+\frac{2}{N_1}\delta_0^{'}C^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{Q}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon\\&+\frac{1}{N_1}\left(\epsilon^{'}\mathbf{G}^{-1^{'}}(\zeta_0)\mathbf{Q}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon-\sigma_{\epsilon0}^2\mathrm{tr}(\mathbf{G}^{-1^{'}}(\zeta_0)\mathbf{Q}(\zeta)\mathbf{G}^{-1}(\zeta_0))\right)\end{aligned}$$

By Lemma A.1 and A.2, $\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{Q}(\zeta)\mathbf{G}^{-1}(\zeta_0)$ and $\mathbf{G}^{-1^{'}}(\zeta_0)\mathbf{Q}(\zeta)\mathbf{G}^{-1}(\zeta_0)$ are uniformly bounded in row and column sum norms. Thus, the first two terms on r.h.s. are point-wise convergent by Lemma A.3(v), and the last term is point-wise convergent by Lemma A.3(iv).

To prove the stochastic equicontinuity, note for any two parameter vectors $\zeta_1,\zeta_2\in\Delta$, it follows from the mean value theorem that

$$\frac{1}{N_1}\left(y^{'}\mathbf{Q}(\zeta_1)y-y^{'}\mathbf{Q}(\zeta_2)y\right)=\frac{1}{N_1}y^{'}\frac{\partial\mathbf{Q}(\bar{\zeta})}{\partial\zeta^{'}}y(\zeta_1-\zeta_2),$$

where $\bar{\zeta}$ is between ζ_1 and ζ_2 elementwise. Thus we need to prove that $\sup_{\zeta\in\Delta}\frac{1}{N_1}y^{'}\frac{\partial\mathbf{Q}(\zeta)}{\partial\alpha}y=O_p(1)$ and $\sup_{\zeta\in\Delta}\frac{1}{N_1}y^{'}\frac{\partial\mathbf{Q}(\zeta)}{\partial\tau}y=O_p(1)$. We will prove the latter and the former can be proved in a similar way. First note

$$\begin{aligned}\frac{\partial\mathbf{Q}(\zeta)}{\partial\tau}&=\mathbf{G}^{'}(\zeta)\mathbf{M}^{'}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)+\mathbf{G}^{'}(\zeta)\dot{\mathbb{Q}}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\\&+\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\dot{\mathbb{Q}}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)+\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\dot{\mathbb{Q}}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\\&+\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\mathbf{G}(\zeta),\end{aligned}$$

where $\dot{\mathbb{Q}}_{\mathbb{C}}(\tau)=\frac{\partial\mathbb{Q}_{\mathbb{C}}(\tau)}{\partial\tau}=-\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\mathbb{P}_{\mathbb{C}}(\tau)+\mathbb{P}_{\mathbb{C}}(\tau)\mathbf{M}'\mathbb{Q}_{\mathbb{C}}(\tau)\right)$ and $\dot{\mathbb{Q}}_{\mathbb{X}}(\tau)=\frac{\partial\mathbb{Q}_{\mathbb{X}}(\tau)}{\partial\tau}$. After some algebra $\frac{\partial\mathbb{X}(\tau)}{\partial\tau}=\frac{\partial\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}X}{\partial\tau}=\mathbb{D}(\tau)\mathbb{X}(\tau)$, where $\mathbb{D}(\tau)=\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}-\mathbb{P}_{\mathbb{C}}(\tau)\mathbf{M}^{'}$. This leads to

$$\dot{\mathbb{Q}}_{\mathbb{X}}(\tau)=\frac{\partial\mathbb{Q}_{\mathbb{X}}(\tau)}{\partial\tau}=-\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{D}(\tau)\mathbb{P}_{\mathbb{X}}(\tau)-\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{D}^{'}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau).$$

Recall that $\phi=X\beta_0+C\delta_0$. Then $y=\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}(\phi+\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)$. Denote $\mathbb{Q}^{\dagger}(\zeta)=\mathbb{Q}^{'}(\zeta)\mathbb{D}(\zeta)\mathbb{Q}(\zeta)$ and $\mathbb{Q}(\zeta)=\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)$. Under our assumptions, Lemmas A.1 and A.2 ensure that $\mathbb{Q}^{\dagger}(\zeta)$ is bounded in row and column sum norms. This leads to

$$\begin{aligned}&y^{'}\frac{\partial\mathbf{Q}(\zeta)}{\partial\tau}y=-2y^{'}\mathbb{Q}^{\dagger}(\zeta)y=-\frac{2}{N_1}(\phi+\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbb{Q}^{\dagger}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}(\phi+\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)\\&=-\frac{2}{N_1}\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbb{Q}^{\dagger}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi-\frac{4}{N_1}\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbb{Q}^{\dagger}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon-\frac{2}{N_1}\epsilon^{'}\mathbf{G}^{-1}(\zeta_0)\mathbb{Q}^{\dagger}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon\\&=O_p(1),\end{aligned}$$

uniformly in $\zeta \in \Delta$ by Lemma A.3. Thus, it follows that $\sup_{\zeta\in\Delta} \frac{1}{N_1} y' \frac{\partial \mathbf{Q}(\zeta)}{\partial \tau} y = O_p(1)$.

Proof of (iii). Using $\bar{\epsilon}(\zeta) = \mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(y - \mathrm{E}(y)\right) + \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y$ in (3.13) and $\widehat{\epsilon}(\zeta) = \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y$ from the proof of (ii), we have

$$
\begin{aligned}
&\frac{1}{N_1} y' \mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\widehat{\epsilon}(\zeta) - \frac{1}{N_1}\mathrm{E}\left(y' \mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\bar{\epsilon}(\zeta)\right) \\
&= \frac{1}{N_1}\left(y' \mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\mathbb{Q}(\zeta)y - \mathrm{E}(y' \mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\mathbb{Q}(\zeta)y)\right) \\
&\quad - \frac{\sigma^2_{\epsilon 0}}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\mathbb{P}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right),
\end{aligned}
$$

where $\mathbb{P}(\zeta) = \mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)$. The uniform convergence of the first term can be shown similar to that of $\frac{1}{N_1}\left(y'\mathbf{Q}(\zeta)y - \mathrm{E}(y'\mathbf{Q}(\zeta)y)\right)$ in the proof of part (ii), and thus is omitted. By Lemma A.5, the second term is $o(1)$ uniformly in $\zeta \in \Delta$.

Proof of (iv). Using the expressions for $\bar{\epsilon}(\zeta)$ and $\widehat{\epsilon}(\zeta)$ from the proof of (iii) again, we have

$$
\begin{aligned}
&\frac{1}{N_1}\widehat{\epsilon}'(\zeta)\mathbf{M}\widehat{\epsilon}(\zeta) - \frac{1}{N_1}\mathrm{E}\left(\bar{\epsilon}'(\zeta)\mathbf{M}\bar{\epsilon}(\zeta)\right) \\
&= \frac{1}{N_1}\left(y'\mathbb{Q}'(\zeta)\mathbf{M}\mathbb{Q}(\zeta)y - \mathrm{E}(y'\mathbb{Q}'(\zeta)\mathbf{M}\mathbb{Q}(\zeta)y)\right) - \frac{\sigma^2_{\epsilon 0}}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbb{P}'(\zeta)\mathbf{M}\mathbb{P}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right) \\
&\quad - \frac{\sigma^2_{\epsilon 0}}{N_1}\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbb{P}'(\zeta)\mathbf{M}^s\mathbb{Q}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right),
\end{aligned}
$$

where $\mathbf{M}^s = \mathbf{M} + \mathbf{M}'$. The uniform convergence of the first term can be shown similar to that of $\frac{1}{N_1}\left(y'\mathbf{Q}(\zeta)y - \mathrm{E}(y'\mathbf{Q}(\zeta)y)\right)$ in the proof of (ii) and thus is omitted. By Lemma A.5, the second and third term are $o(1)$ uniformly in $\zeta \in \Delta$.

C.2 Proof of Theorem 3.2

The mean value theorem gives $\sqrt{N_1}(\widehat{\theta}^* - \theta_0) = -\left(\frac{1}{N_1}\frac{\partial S^*(\bar{\theta})}{\partial \theta'}\right)^{-1}\frac{1}{\sqrt{N_1}}S^*(\theta_0)$, where $\bar{\theta}$ is between $\widehat{\theta}^*$ and θ_0 elementwise (Jennrich, 1969, Lemma 3). We need to prove the following results:

(i) $\frac{1}{\sqrt{N_1}}S^*(\theta_0) \xrightarrow{d} N\left[0, \lim_{N\to\infty}\Omega^*(\theta_0)\right]$,

(ii) $\frac{1}{N_1}\left(\frac{\partial S^*(\bar{\theta})}{\partial \theta'} - \frac{\partial S^*(\theta_0)}{\partial \theta'}\right) = o_p(1)$,

(iii) $\frac{1}{N_1}\left(\frac{\partial S^*(\theta_0)}{\partial \theta'} - \mathrm{E}\left(\frac{\partial S^*(\theta_0)}{\partial \theta'}\right)\right) = o_p(1)$.

Proof of (i). Note that elements of $S^*(\theta_0)$ in (3.16) are linear-quadratic forms in ϵ. Let $a = (a_1', a_2, a_3, a_4)'$ for an $k \times 1$ constant vector a_1 and constants a_2, a_3 and a_4. Then we can express $a'S^*(\theta_0)$ as $b'\epsilon + \epsilon' B\epsilon - \sigma^2_{\epsilon 0}\mathrm{tr}(B)$, where $b' = \frac{1}{\sigma^2_{\epsilon 0}}a_1'\mathbb{X}'(\tau_0) - \frac{a_3}{\sigma^2_{\epsilon 0}}\phi'\mathbf{e}^{\tau_0\mathbf{M}'}\mathbf{S}'(\tau_0)\mathbb{Q}_{\mathbb{C}}(\tau_0)$ and $B = \frac{a_2}{2\sigma^2_{\epsilon 0}}\mathbb{Q}_{\mathbb{C}}(\tau_0) - \frac{a_3}{\sigma^2_{\epsilon 0}}\mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{S}(\tau_0) - \frac{a_4}{\sigma^2_{\epsilon 0}}\mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M}\mathbb{Q}_{\mathbb{C}}(\tau_0)$. Since b and B satisfy the conditions for the CLT in Kelejian and Prucha (2001) by Lemma A.1 and A.2(i), $\frac{1}{N_1}a'S^*(\theta_0)$ is asymptotically normal. Thus, the Cramér-Wold device leads to (i).

Proof of (ii). The explicit expressions of the elements of the Hessian matrix $H^*(\theta)$ are given in the main paper. By Assumptions 3-4, $\mathbf{S}(\tau)$ is bounded in row sum and column sum matrix norms. Since $y = \mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}(\phi + \mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)$, all terms in the Hessian matrix can be written in forms of functions in Lemma A.3, and thus $\frac{1}{N_1}H^*(\theta_0) = O_p(1)$. As $\bar{\sigma}_\epsilon^2 \xrightarrow{p} \sigma_{\epsilon 0}^2$, $\bar{\sigma}_\epsilon^{-r} = \bar{\sigma}_{\epsilon 0}^{-r} + o_p(1)$ for $r = 2, 4, 6$. Note σ_ϵ^r appears in $H^*(\theta)$ multiplicatively, which implies $\frac{1}{N_1}H^*(\bar{\beta}, \bar{\sigma}_\epsilon^2, \bar{\alpha}, \bar{\tau}) = \frac{1}{N_1}H^*(\bar{\beta}, \sigma_{\epsilon 0}^2, \bar{\alpha}, \bar{\tau}) + o_p(1)$, where an error appears that can be neglected asymptotically. Then the proof of (ii) is equivalent to the proof of

$$\frac{1}{N_1}\left(H^*(\bar{\beta}, \sigma_{\epsilon 0}^2, \bar{\alpha}, \bar{\tau}) - H^*(\theta_0)\right) \xrightarrow{p} 0.$$

We first consider the random elements in $H^*(\theta)$. We can write $\mathbf{e}^{\bar{\boldsymbol{\alpha}}\mathbf{W}} = (\mathbf{e}^{\bar{\boldsymbol{\alpha}}\mathbf{W}} - \mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}) + \mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}$, $\mathbf{e}^{\bar{\boldsymbol{\tau}}\mathbf{M}} = (\mathbf{e}^{\bar{\boldsymbol{\tau}}\mathbf{M}} - \mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}) + \mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}$ and $\bar{\beta} = (\bar{\beta} - \beta_0) + \beta_0$, and then expand the terms in $\frac{1}{N_1}H^*(\theta)$. By Lemma A.2 and A.3 and the reduced form of y, $\frac{1}{N_1}y'Ay = O_p(1)$ and $\frac{1}{N_1}X'Ay = O_p(1)$, where A is an $N \times N$ matrix that is uniformly bounded. Also note $\left\|\mathbf{e}^{\bar{\boldsymbol{\alpha}}\mathbf{W}} - \mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}\right\|_\infty = \left\|(\mathbf{e}^{(\bar{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)\mathbf{W}} - I_N)\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}\right\|_\infty \leq \left\|(\mathbf{e}^{(\bar{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)\mathbf{W}} - I_N\right\|_\infty \left\|\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}\right\|_\infty = o_p(1)$ by Lemma A.4, and similarly $\left\|\mathbf{e}^{\bar{\boldsymbol{\tau}}\mathbf{M}} - \mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}\right\|_\infty = o_p(1)$. Then from the expanded forms of the random elements of $\frac{1}{N_1}\left(H^*(\bar{\beta}, \sigma_{\epsilon 0}^2, \bar{\alpha}, \bar{\tau}) - H^*(\theta_0)\right)$, we infer that these elements are $o_p(1)$. For the nonrandom, i.e., the trace terms in $\frac{1}{N_1}\left(H^*(\bar{\beta}, \sigma_{\epsilon 0}^2, \bar{\alpha}, \bar{\tau}) - H^*(\theta_0)\right)$, the convergence results follow from the continuous mapping theorem (see Proposition 2.27 in White (2001)) since $\bar{\tau} - \tau_0 = o_p(1)$.

Proof of (iii). Note each element of $\frac{1}{N_1}\left(\frac{\partial S^*(\theta_0)}{\partial \theta'} - \mathrm{E}\left(\frac{\partial S^*(\theta_0)}{\partial \theta'}\right)\right)$ is a linear or quadratic function of ϵ by the reduced form of y. By Lemma A.3, $\frac{1}{N_1}\left(\frac{\partial S^*(\theta_0)}{\partial \theta'} - \mathrm{E}\left(\frac{\partial S^*(\theta_0)}{\partial \theta'}\right)\right) = o_p(1)$.

C.3 Proof of Theorem 3.3

Since $\widehat{\theta}^*$, $\widehat{\rho}$ and $\widehat{\kappa}$ are consistent, substituting them into $\Omega^*(\theta)$ does not cause any bias as an estimator for $\Omega^*(\theta_0)$. For $\widehat{\delta}^*$, however, the incidental parameter problem makes it inconsistent when T is fixed, which leads to bias. The bias is derived as following. From (3.3) in the main paper, $\widehat{\delta}(\beta, \zeta) = \left(\mathbb{C}'(\tau)\mathbb{C}(\tau)\right)^{-1}\mathbb{C}'(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y - X\beta)$. Also note $\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^*\mathbf{W}}y - X\widehat{\beta}^* = \mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}y - X\beta_0 + (\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^*\mathbf{W}} - \mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}})y - X(\widehat{\beta}^* - \beta_0)$ and $\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}y - X\beta_0 = C\delta_0 + \mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon$. Let $\widehat{\delta}^* = \widehat{\delta}(\widehat{\beta}^*, \widehat{\zeta}^*)$. Applying the mean value theorem to $C\widehat{\delta}^*$ with respect to the $\widehat{\tau}^*$ element, we have

$$\begin{aligned}
C\widehat{\delta}^* &= \mathbf{e}^{-\widehat{\boldsymbol{\tau}}^*\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\widehat{\tau}^*)\mathbf{e}^{\widehat{\boldsymbol{\tau}}^*\mathbf{M}}(\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^*\mathbf{W}}y - X\widehat{\beta}^*) \\
&= \left(\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau_0)\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}} + \xi(\bar{\tau})(\widehat{\tau}^* - \tau_0)\right)(\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^*\mathbf{W}}y - X\widehat{\beta}^*) \\
&= C\delta_0 + \mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau_0)\epsilon + \mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau_0)\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}\left((\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^*\mathbf{W}} - \mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}})y - X(\widehat{\beta}^* - \beta_0)\right) \\
&\quad + \xi(\bar{\tau})(\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^*\mathbf{W}}y - X\widehat{\beta}^*)(\widehat{\tau}^* - \tau_0),
\end{aligned} \tag{C.3}$$

where $\bar{\tau}$ lies between $\widehat{\tau}^*$ and τ_0 and $\xi(\bar{\tau}) = \frac{\partial \mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}}{\partial \tau}|_{\tau = \bar{\tau}} = \mathbf{e}^{-\bar{\boldsymbol{\tau}}\mathbf{M}}(\mathbb{D}(\bar{\tau})\mathbb{Q}_{\mathbb{C}}(\bar{\tau}) - \mathbf{M})\mathbf{e}^{\bar{\boldsymbol{\tau}}\mathbf{M}} - \mathbf{M}\mathbf{e}^{-\bar{\boldsymbol{\tau}}\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\bar{\tau})\mathbf{e}^{\bar{\boldsymbol{\tau}}\mathbf{M}}$.

Note $\left\|\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^*\mathbf{W}} - \mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}\right\|_\infty = \left\|(\mathbf{e}^{(\widehat{\boldsymbol{\alpha}}^* - \boldsymbol{\alpha}_0)\mathbf{W}} - I_N)\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}\right\|_\infty \leq \left\|\mathbf{e}^{(\widehat{\boldsymbol{\alpha}}^* - \boldsymbol{\alpha}_0)\mathbf{W}} - I_N\right\|_\infty \left\|\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}\right\|_\infty =$

$o_p(1)$ by Lemma A.4. Using Lemmas A.3 and A.4, we can express the terms in $\Omega^*(\widehat{\theta}^*)$ that are linear in $C\widehat{\delta}^*$ in the following way:

$$\frac{1}{N_1}a^{'}C\widehat{\delta}^* = \frac{1}{N_1}a^{'}C\delta_0 + \frac{1}{N_1}a^{'}\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau_0)\epsilon + o_p(1) = \frac{1}{N_1}a^{'}C\delta_0 + o_p(1), \tag{C.4}$$

where a is a suitable vector. Thus, the terms that are linear in $C\widehat{\delta}^*$ can be consistently estimated. The only quadratic term in $\phi = X\beta_0 + C\delta_0$ is $\frac{1}{N_1\sigma^2_{\epsilon 0}}\phi^{'}\mathcal{Q}_3^{'}(\tau_0)\mathcal{Q}_3(\tau_0)\phi$, which is contained in $\Omega^*_{\alpha\alpha}(\theta_0)$. Then, the only quadratic term in δ_0 is $\frac{1}{N_1\sigma^2_{\epsilon 0}}\delta_0^{'}C^{'}\mathcal{Q}_3^{'}(\tau_0)\mathcal{Q}_3(\tau_0)C\delta_0$. Recall that $\mathcal{Q}_2(\tau) = \mathcal{Q}_3(\tau)\mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}}$. Then, using (C.3), we obtain

$$\begin{aligned}
&\frac{1}{N_1\widehat{\sigma}^{*2}_\epsilon}\widehat{\delta}^{*'}C^{'}\mathcal{Q}_3^{'}(\widehat{\tau}^*)\mathcal{Q}_3(\widehat{\tau}^*)C\widehat{\delta}^* \\
&= \frac{1}{N_1\widehat{\sigma}^{*2}_\epsilon}\delta_0^{'}C^{'}\mathcal{Q}_3^{'}(\widehat{\tau}^*)\mathcal{Q}_3(\widehat{\tau}^*)C\delta_0 + \frac{1}{N_1\widehat{\sigma}^{*2}_\epsilon}\epsilon^{'}\mathbb{P}_{\mathbb{C}}(\tau_0)\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}^{'}}\mathcal{Q}_3^{'}(\tau_0)\mathcal{Q}_3(\tau_0)\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau_0)\epsilon + o_p(1) \\
&= \frac{1}{N_1\sigma^2_{\epsilon 0}}\delta_0^{'}C^{'}\mathcal{Q}_3^{'}(\tau_0)\mathcal{Q}_3(\tau_0)C\delta_0 + \frac{1}{N_1}\mathrm{tr}\left(\mathbb{P}_{\mathbb{C}}(\tau_0)\mathcal{Q}_2^{'}(\tau_0)\mathcal{Q}_2(\tau_0)\right) + o_p(1).
\end{aligned} \tag{C.5}$$

Thus, the bias term, which has the same dimension of $\Omega^*(\theta_0)$, is a matrix of zeros except the (α,α) element, which is $\frac{1}{N_1}\mathrm{tr}\left(\mathbb{P}_{\mathbb{C}}(\tau_0)\mathcal{Q}_2^{'}(\tau_0)\mathcal{Q}_2(\tau_0)\right)$. Since the bias term only involves τ_0, we can formulate a consistent estimator based on the plug-in estimator by the continuous mapping theorem.

C.4 Proof of Theorem 3.4

Let $\{\epsilon_j\}$, $\{\tilde{\epsilon}_j\}$ and $\{\widehat{\epsilon}_j\}$ be the jth element of ϵ, $\tilde{\epsilon} = \mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon$ and $\widehat{\epsilon} = \mathbb{Q}_{\mathbb{C}}(\widehat{\tau}^*)\mathbf{e}^{\widehat{\boldsymbol{\tau}}^*\mathbf{M}}(\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^*\mathbf{W}}y - X\widehat{\beta}^*)$ respectively for $j = 1,\dots,N$. Let q_{jh} be the (j,h)th element of $\mathbb{Q}_{\mathbb{C}}(\tau_0)$ for $j,h = 1,\dots,N$.
To prove the consistency of $\widehat{\rho}$, note that $\widehat{\sigma}^* - \sigma_{\epsilon 0} = o_p(1)$ and $\widehat{\tau}^* - \tau_0 = o_p(1)$, thus the denominator of $\widehat{\rho}$ is consistent, i.e., it converges in probability to its population counterpart. It's left to prove that the numerator of $\widehat{\rho}$ is consistent, i.e., $\frac{1}{N}\sum_{j=1}^N(\widehat{\epsilon}_j^3 - \mathrm{E}(\tilde{\epsilon}_j^3)) = 0$ or equivalently,

(1) $\frac{1}{N}\sum_{j=1}^N(\widehat{\epsilon}_j^3 - \tilde{\epsilon}_j^3) \xrightarrow{p} 0$,

(2) $\frac{1}{N}\sum_{j=1}^N\left(\tilde{\epsilon}_j^3 - \mathrm{E}(\tilde{\epsilon}_j^3)\right) \xrightarrow{p} 0$.

Proof of (1). For simplicity of exposition, we denote $\omega = (\beta^{'},\zeta^{'})^{'}$ and write $\tilde{\epsilon}(\beta,\zeta)$ as $\tilde{\epsilon}(\omega) = \mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y - X\beta)$. Let $\chi(\omega) = \frac{\partial\tilde{\epsilon}(\omega)}{\partial\omega}$, then

$$\chi(\omega) = \begin{pmatrix} -\mathbb{X}(\tau) & y(\zeta) & (\dot{\mathbb{Q}}_{\mathbb{C}}(\tau) + \mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M})\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y - X\beta)\end{pmatrix}. \tag{C.6}$$

Let $\chi_j(\omega)$ be the jth row of $\chi(\omega)$. Then, by the MVT,

$$\widehat{\epsilon}_j \equiv \tilde{\epsilon}_j(\widehat{\omega}^*) = \tilde{\epsilon}_j(\omega_0) + \chi_j^{'}(\bar{\omega})(\widehat{\omega}^* - \omega_0) = \tilde{\epsilon}_j(\omega_0) + \nu_j^{'}(\widehat{\omega}^* - \omega_0) + o_p(\|\widehat{\omega}^* - \omega_0\|), \tag{C.7}$$

for each $j = 1,\dots,N$, where $\bar{\omega}$ is between $\widehat{\omega}^*$ and ω_0 elementwise and $\nu_j^{'} = \mathrm{plim}_{N\to\infty}\chi_j^{'}(\bar{\omega})$. We will prove that $\nu_j^{'} = O_p(1)$ below. We start from the first k elements of $\nu_j^{'}$, i.e., $-\mathrm{plim}\,\mathbb{X}(\bar{\tau})$.

Since $\bar{\tau} \xrightarrow{p} \tau_0$ by $\widehat{\tau}^* \xrightarrow{p} \tau_0$, we know $-\operatorname{plim} \mathbb{X}(\bar{\tau}) = -\mathbb{X}(\tau_0)$, which indicates that the first k elements are $O(1)$. Similarly, the $(k+1)$th and $(k+2)$th elements of $\nu_j^{'}$ are $y(\zeta_0)$ and $(\dot{\mathbb{Q}}_{\mathbb{C}}(\tau_0) + \mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M})\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}(\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}y - X\beta_0)$ respectively. Substituting $y = \mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi + \mathbf{G}^{-1}(\zeta_0)\epsilon$ into these elements leads to $y(\zeta_0) = \mathcal{Q}_3(\tau_0)\phi - \mathcal{Q}_2(\tau_0)\epsilon$ and $(\dot{\mathbb{Q}}_{\mathbb{C}}(\tau_0) + \mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M})\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}(\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}y - X\beta_0) = (\dot{\mathbb{Q}}_{\mathbb{C}}(\tau_0) + \mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M})\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}C\delta_0 + (\dot{\mathbb{Q}}_{\mathbb{C}}(\tau_0) + \mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M})\epsilon$. By Lemmas A.1 and A.2, and Assumptions 3-4, the elements of $\mathcal{Q}_3(\tau_0)\phi$ and $(\dot{\mathbb{Q}}_{\mathbb{C}}(\tau_0) + \mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M})\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}C\delta_0$ are uniformly bounded, and $\mathcal{Q}_2(\tau_0)$ and $\dot{\mathbb{Q}}_{\mathbb{C}}(\tau_0) + \mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M}$ are uniformly bounded in both row sum and column sum matrix norms. Therefore each element of $y(\zeta_0)$ and $(\dot{\mathbb{Q}}_{\mathbb{C}}(\tau_0) + \mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M})\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}(\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}y - X\beta_0)$ is $O_p(1)$, i.e., the $(k+1)$th and $(k+2)$th elements of $\nu_j^{'}$ are $O_p(1)$ for $j = 1, \ldots, N$.

Note that we can write

$$\tilde{\epsilon}_j = \sum_{h=1}^{N} q_{jh}\epsilon_h. \tag{C.8}$$

Since $\tilde{\epsilon} = O_p(1)$, $\chi_j^{'} = O_p(1)$ and $\widehat{\omega}^* - \omega_0 = O_p(\frac{1}{\sqrt{N_1}})$, also by the fact that ϵ_j is *i.i.d* for $j = 1, \ldots, N$, we have $\widehat{\epsilon}_j^3 = \tilde{\epsilon}_j^3 + 3\tilde{\epsilon}_j^2\nu_j^{'}(\widehat{\omega}^* - \omega_0) + o_p(\|\widehat{\omega}^* - \omega_0\|)$. Using (C.8), we have

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N}\left(\widehat{\epsilon}_j^3 - \tilde{\epsilon}_j^3\right) &= \frac{3}{N}\sum_{j=1}^{N}\tilde{\epsilon}_j^2\nu_j^{'}(\widehat{\omega}^* - \omega_0) + o_p(\|\widehat{\omega}^* - \omega_0\|) \\
&= \frac{3\sigma_{\epsilon 0}^2}{N}\sum_{j=1}^{N}(\sum_{h=1}^{N} q_{jh}^2\nu_j^{'})(\widehat{\omega}^* - \omega_0) + o_p(\|\widehat{\omega}^* - \omega_0\|) \\
&= o_p(1),
\end{aligned}$$

since $\frac{1}{N}\sum_{j=1}^{N}\nu_j^{'} = O_p(1)$.

Proof of (2). Substituting (C.8) into the function in (2) gives

$$\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N}\left(\tilde{\epsilon}_j^3 - \mathrm{E}(\tilde{\epsilon}_j^3)\right) = &\frac{1}{N}\sum_{j=1}^{N}\sum_{h=1}^{N} q_{jh}^3\left(\epsilon_h^3 - \mathrm{E}(\epsilon_h^3)\right) + \frac{3}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1\\ m\neq l}}^{N} q_{jl}^2 q_{jm}\epsilon_l^2\epsilon_m \\
&+ \frac{6}{N}\sum_{j=1}^{N}\sum_{m=1}^{N}\sum_{\substack{l=1\\ l\neq m}}^{N}\sum_{\substack{h=1\\ h\neq m,l}}^{N} q_{jm}q_{jl}q_{jh}\epsilon_m\epsilon_l\epsilon_h.
\end{aligned} \tag{C.9}$$

For the first term, by Lemma A.2, the elements of $\mathbb{Q}_{\mathbb{C}}(\tau_0)$ are uniformly bounded, i.e., there exists a constant $\bar{q}$ such that $|q_{jh}| \leq \bar{q}$ for all j and h. Thus $\sum_{j=1}^{N} q_{jh}^3 \leq \bar{q}^2\sum_{j=1}^{N}|q_{jh}| < \infty$. Since $\{\epsilon_i\}$ are *i.i.d*, by the Weak Law of Large Numbers, the first term converge to 0 in probability as $N \longrightarrow \infty$.

The second term can be expressed as

$$\frac{3}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1\\m\neq l}}^{N} q_{jl}^2 q_{jm}\epsilon_l^2\epsilon_m = \frac{3}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1\\m\neq l}}^{N} q_{jl}^2 q_{jm}(\epsilon_l^2-\sigma_{\epsilon 0}^2)\epsilon_m + \frac{3}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1\\m\neq l}}^{N} q_{jl}^2 q_{jm}\sigma_{\epsilon 0}^2\epsilon_m$$

$$= \frac{3}{N}\sum_{m=1}^{N}(\epsilon_m^2-\sigma_{\epsilon 0}^2)(\sum_{j=1}^{N}\sum_{l=1}^{m-1} q_{jm}^2 q_{jl}\epsilon_l) + \frac{3}{N}\sum_{m=1}^{N}\epsilon_m\left(\sum_{j=1}^{N}\sum_{l=1}^{m-1} q_{jl}^2 q_{jm}(\epsilon_l^2-\sigma_{\epsilon 0}^2)\right)$$

$$+ \frac{3}{N}\sum_{m=1}^{N}\sum_{j=1}^{N}\sum_{\substack{l=1\\l\neq m}}^{N} q_{jl}^2 q_{jm}\sigma_{\epsilon 0}^2\epsilon_m$$

$$= \frac{3}{N}\sum_{m=1}^{N}(h_{1,m}+h_{2,m}+h_{3,m}), \tag{C.10}$$

where $h_{1,m} = (\epsilon_m^2-\sigma_{\epsilon 0}^2)(\sum_{j=1}^{N}\sum_{l=1}^{m-1} q_{jm}^2 q_{jl}\epsilon_l)$, $h_{2,m} = \epsilon_m(\sum_{j=1}^{N}\sum_{l=1}^{m-1} q_{jl}^2 q_{jm}(\epsilon_l^2-\sigma_{\epsilon 0}^2))$ and $h_{3,m} = \sum_{j=1}^{N}\sum_{\substack{l=1\\l\neq m}}^{N} q_{jl}^2 q_{jm}\sigma_{\epsilon 0}^2\epsilon_m$.

Finally, the third term can be expressed as

$$\frac{6}{N}\sum_{j=1}^{N}\sum_{m=1}^{N}\sum_{\substack{l=1\\l\neq m}}^{N}\sum_{\substack{h=1\\h\neq m,l}}^{N} q_{jm}q_{jl}q_{jh}\epsilon_m\epsilon_l\epsilon_h = \frac{18}{N}\sum_{m=1}^{N}\epsilon_m(\sum_{j=1}^{N}\sum_{l=1}^{m-1}\sum_{\substack{h=1\\h\neq l}}^{m-1} q_{jm}q_{jl}q_{jh}\epsilon_l\epsilon_h) = \frac{18}{N}\sum_{m=1}^{N} h_{4,m} \tag{C.11}$$

where $h_{4,m} = \epsilon_m(\sum_{j=1}^{N}\sum_{l=1}^{m-1}\sum_{\substack{h=1\\h\neq l}}^{m-1} q_{jm}q_{jl}q_{jh}\epsilon_l\epsilon_h)$. Let $\{\mathcal{F}_m\}$ be the increasing sequence of σ-fields generated by $(\epsilon_1,\ldots,\epsilon_j, j=1,\ldots,m)$ for $m=1,\ldots,N$. Then $\mathrm{E}[(h_{1,m},h_{2,m},h_{3,m},h_{4,m})|\mathcal{F}_{m-1}] = 0$, i.e., $\{(h_{1,m},h_{2,m},h_{3,m},h_{4,m})',\mathcal{F}_m\}$ form a vector martingale difference (M.D.) sequence. By Assumption 1 and the fact that $\mathbb{Q}_{\mathbb{C}}(\tau_0)$ is bounded in both row sum and column sum matrix norms, $\mathrm{E}\,|h_{r,m}|^{1+\varrho} < \infty$ for $r=1,2,3,4$ and $\varrho > 0$. Hence $\{h_{1,m}\}$, $\{h_{2,m}\}$, $\{h_{3,m}\}$ and $\{h_{4,m}\}$ are uniformly integrable. By Theorem 19.7 in Davidson (1994), the second and third term converge to 0 in probability.

To prove the consistency of $\widehat{\kappa}$, similar to the proof for the consistency of $\widehat{\rho}$, we need to show

(3) $\frac{1}{N}\sum_{j=1}^{N}\left(\widehat{\epsilon}_j^4 - \tilde{\epsilon}_j^4\right) \xrightarrow{p} 0$,

(4) $\frac{1}{N}\sum_{j=1}^{N}\left(\tilde{\epsilon}_j^4 - \mathrm{E}(\tilde{\epsilon}_j^4)\right) \xrightarrow{p} 0$.

Proof of (3). Using (C.7), we first write $\widehat{\epsilon}_j^4 = \tilde{\epsilon}_j^4 + 4\tilde{\epsilon}_j^3\nu_j'(\widehat{\omega}^* - \omega_0) + o_p(\|\widehat{\omega}^* - \omega_0\|)$. Summing over

j for $j = 1, \ldots, N$, we have

$$
\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N}\left(\widehat{\epsilon}_j^4 - \tilde{\epsilon}_j^4\right) &= \frac{4}{N}\sum_{j=1}^{N}\tilde{\epsilon}_j^3 \nu_j'(\widehat{\omega}^* - \omega_0) + o_p(\|\widehat{\omega}^* - \omega_0\|) \\
&= \frac{4\sigma_{\epsilon 0}^3}{N}\sum_{j=1}^{N}(\sum_{h=1}^{N} q_{jh}^3 \nu_j')(\widehat{\omega}^* - \omega_0) + o_p(\|\widehat{\omega}^* - \omega_0\|) \\
&= o_p(1),
\end{aligned}
$$

since $\frac{1}{N}\sum_{j=1}^{N}\nu_j' = O_p(1)$ as shown in the proof of (1).

Proof of (4). Using (C.8) we write

$$
\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N}\left(\tilde{\epsilon}_j^4 - \mathrm{E}(\tilde{\epsilon}_j^4)\right) = {} & \frac{1}{N}\sum_{j=1}^{N}\sum_{h=1}^{N} q_{jh}^4\left(\epsilon_h^4 - \mathrm{E}(\epsilon_h^4)\right) + \frac{3}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1\\ m\neq l}}^{N} q_{jl}^2 q_{jm}^2(\epsilon_l^2\epsilon_m^2 - \sigma_{\epsilon 0}^4) \\
&+ \frac{4}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1\\ m\neq l}}^{N} q_{jl}^3 q_{jm}\epsilon_l^3\epsilon_m + \frac{6}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1\\ m\neq l}}^{N}\sum_{\substack{h=1\\ h\neq m,l}}^{N} q_{jl}^2 q_{jm} q_{jh}\epsilon_l^2\epsilon_m\epsilon_h \\
&+ \frac{1}{N}\sum_{j=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1\\ m\neq l}}^{N}\sum_{\substack{h=1\\ h\neq m,l}}^{N}\sum_{\substack{p=1\\ p\neq m,l,h}}^{N} q_{jl}q_{jm}q_{jh}q_{jp}\epsilon_l\epsilon_m\epsilon_h\epsilon_p. \qquad \text{(C.12)}
\end{aligned}
$$

The proofs for the first, third, fourth and fifth term are similar to those in the proof of (2) and thus are omitted. For the second term, we can write $\epsilon_l^2\epsilon_m^2 - \sigma_{\epsilon 0}^4 = (\epsilon_l^2 - \sigma_{\epsilon 0}^2)(\epsilon_m^2 - \sigma_{\epsilon 0}^2) + \sigma_{\epsilon 0}^2(\epsilon_m^2 - \sigma_{\epsilon 0}^2) + \sigma_{\epsilon 0}^2(\epsilon_l^2 - \sigma_{\epsilon 0}^2)$. Then the second term equals

$$
\begin{aligned}
&\frac{6}{N}\sum_{l=1}^{N}(\epsilon_l^2 - \sigma_{\epsilon 0}^2)\left(\sum_{j=1}^{N}\sum_{m=1}^{l-1} q_{jl}^2 q_{jm}^2(\epsilon_m^2 - \sigma_{\epsilon 0}^2)\right) + \frac{6}{N}\sum_{l=1}^{N}\left(\sum_{j=1}^{N}\sum_{\substack{m=1\\ m\neq l}}^{N} q_{jl}^2 q_{jm}^2\sigma_{\epsilon 0}^2(\epsilon_l^2 - \sigma_{\epsilon 0}^2)\right) \\
&\equiv \frac{6}{N}\sum_{l=1}^{N}(g_{1,l} + g_{2,l}), \qquad \text{(C.13)}
\end{aligned}
$$

where $g_{1,l} = (\epsilon_l^2 - \sigma_{\epsilon 0}^2)\sum_{j=1}^{N}\sum_{m=1}^{l-1} q_{jl}^2 q_{jm}^2(\epsilon_m^2 - \sigma_{\epsilon 0}^2)$ and $g_{2,l} = \sum_{j=1}^{N}\sum_{\substack{m=1\\ m\neq l}}^{N} q_{jl}^2 q_{jm}^2\sigma_{\epsilon 0}^2(\epsilon_l^2 - \sigma_{\epsilon 0}^2)$. Here the summation in the first term in the first equation starts from $l = 2$ but we still write it as starting from $l = 1$ to get the convenient expression at the end. Note $\mathrm{E}[g_{1,l}|\mathcal{F}_{l-1}] = 0$ and $\{g_{2,l}\}$ are independent. Thus they each forms an M.D. sequence. Also $\mathrm{E}\,|g_{r,l}|^{1+\varrho} < \infty$ for $r = 1, 2$ and $\varrho > 0$, thus $\{g_{1,l}\}$ and $\{g_{2,l}\}$ are uniformly integrable. By Theorem 19.7 in Davidson (1994), $\frac{6}{N}\sum_{l=1}^{N} g_{r,l} = o_p(1)$ for $r = 1, 2$.

C.5 Proof of Theorem 4.1

Given Assumption 9, we need to prove $\sup_{\zeta\in\Delta}\frac{1}{N_1}\left\|S^{\dagger c}(\zeta)-\bar{S}^{\dagger c}(\zeta)\right\| \xrightarrow{p} 0$. Let $\mathbb{M}(\tau) = I_N - \mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}X\left(\mathbb{X}'(\tau)\mathbb{X}(\tau)\right)^{-1}\mathbb{X}'(\tau)$ and $\mathbb{N}(\tau) = I_N - \mathbb{M}(\tau)$. Then $\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\left(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y - X\widehat{\beta}^{\dagger}(\zeta)\right) = \mathbb{M}(\tau)\mathbf{G}(\zeta)y$, and $\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\left(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y - X\bar{\beta}^{\dagger}(\zeta)\right) = \mathbb{M}(\tau)\mathbf{G}(\zeta)y + \mathbb{N}(\tau)\mathbf{G}(\zeta)(y-\mathrm{E}(y))$. Also recall that $\widehat{\epsilon}(\zeta) = \tilde{\epsilon}(\widehat{\beta}^{\dagger}(\zeta),\zeta) = \mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\left(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y - X\widehat{\beta}^{\dagger}(\zeta)\right) = \mathbb{Q}(\zeta)y$, where $\mathbb{Q}(\zeta) = \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)$, and

$$\bar{\epsilon}(\zeta) = \mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(y-\mathrm{E}(y)\right) + \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y = \mathbb{P}(\zeta)(y-\mathrm{E}(y)) + \mathbb{Q}(\zeta)y,$$

where $\mathbb{P}(\zeta) = \mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)$. Then the proof of $\sup_{\zeta\in\Delta}\frac{1}{N_1}\left\|S^{\dagger c}(\zeta)-\bar{S}^{\dagger c}(\zeta)\right\| \xrightarrow{p} 0$ is equivalent to the proof of the following results:

(i) $\sup_{\zeta\in\Delta}\frac{1}{N_1}\left\|y'\mathbf{Q}_r^e(\zeta)y - \mathrm{E}\left(y'\mathbf{Q}_r^e(\zeta)y\right)\right\| = o_p(1)$ for $r=1,2$,

(ii) $\sup_{\zeta\in\Delta}\frac{1}{N_1}\mathrm{tr}\left(\Gamma\mathbf{G}^{-1'}(\zeta_0)\mathbf{P}_s^e(\zeta)\mathbf{G}^{-1}(\zeta_0)\right) = o(1)$, for $s=1,2,3,4$,

where $\mathbf{Q}_1^e(\zeta) = \mathbf{G}'(\zeta)\left(\mathbf{S}'(\tau)-\bar{\mathbf{S}}'(\tau)\right)\mathbb{Q}(\zeta)$, $\mathbf{Q}_2^e(\zeta) = \mathbf{G}'(\zeta)\mathbb{M}'(\zeta)\left(\mathcal{Q}_4(\tau)-\bar{\mathcal{Q}}_4(\tau)\right)\mathbb{Q}(\zeta)$, $\mathbf{P}_1^e(\zeta) = \mathbf{G}'(\zeta)\left(\mathbf{S}'(\tau)-\bar{\mathbf{S}}'(\tau)\right)\mathbb{P}(\zeta)$, $\mathbf{P}_2^e(\zeta) = \mathbf{G}'(\zeta)\mathbb{M}'(\tau)\left(\mathcal{Q}_4(\tau)-\bar{\mathcal{Q}}_4(\tau)\right)\mathbb{P}(\zeta)$, $\mathbf{P}_3^e(\zeta) = \mathbf{G}'(\zeta)\mathbb{N}'(\zeta)(\mathcal{Q}_4(\tau)-\bar{\mathcal{Q}}_4(\tau))\mathbb{Q}(\zeta)$, and $\mathbf{P}_4^e(\zeta) = \mathbf{G}'(\zeta)\mathbb{N}'(\zeta)\left(\mathcal{Q}_4(\tau)-\bar{\mathcal{Q}}_4(\tau)\right)\mathbb{P}(\zeta)$.

Proof of (i). Note that $\mathbf{Q}_1^e(\zeta) = \mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\mathbb{Q}(\zeta) - \mathbf{G}'(\zeta)\bar{\mathbf{S}}'(\zeta)\mathbb{Q}(\zeta)$. Since $\bar{\mathbf{S}}'(\zeta)$ is a diagonal matrix, it is bounded in both row sum and column sum matrix norms uniformly in $\zeta\in\Delta$. Then by Lemma A.1, $\mathbf{Q}_1^e(\zeta)$ is uniformly bounded in both row sum and column sum matrix norms uniformly in $\zeta\in\Delta$. Similarly $\mathbf{Q}_2^e(\zeta) = \mathbb{Q}'(\zeta)\mathbf{M}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}(\zeta) - \mathbf{G}'(\zeta)\mathbb{M}'(\zeta)\bar{\mathcal{Q}}_4(\tau)\mathbb{Q}(\zeta)$ is also bounded in both row sum and column sum matrix norms uniformly in $\zeta\in\Delta$. Then $\mathbf{Q}_1^e(\zeta)$ and $\mathbf{Q}_2^e(\zeta)$ have similar forms to $\mathbf{Q}(\zeta)$ in the proof of Theorem 3.1(ii). The proof is similar and thus is omitted.

Proof of (ii). Since $\mathbf{P}_1^e(\zeta)$, $\mathbf{P}_2^e(\zeta)$ and $\mathbf{P}_4^e(\zeta)$ contain $\mathbb{P}(\zeta) = \mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)$, by Lemma A.5, $\sup_{\zeta\in\Delta}\frac{1}{N_1}\mathrm{tr}\left(\Gamma\mathbf{G}'^{-1}(\zeta_0)\mathbf{P}_s^e(\zeta)\mathbf{G}^{-1}(\zeta_0)\right) = o(1)$ for $s=1,2,4$. Recall $\mathbb{Q}(\zeta) = \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)$ and $\mathbb{N}(\tau) = \mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}X\left(\mathbb{X}'(\tau)\mathbb{X}(\tau)\right)^{-1}\mathbb{X}'(\tau)$. Then for $\mathbf{P}_3^e(\zeta)$, we have

$$\begin{aligned}
&\frac{1}{N_1}\mathrm{tr}\left(\Gamma\mathbf{G}^{-1'}(\zeta_0)\mathbf{P}_3^e(\zeta)\mathbf{G}^{-1}(\zeta_0)\right) = \frac{1}{N_1}\mathrm{tr}\left(\mathbf{G}'(\zeta)\mathbb{N}'(\zeta)\left(\mathcal{Q}_4(\tau)-\bar{\mathcal{Q}}_4(\tau)\right)\mathbb{Q}(\zeta)\mathrm{Var}(y)\right)\\
&= \frac{1}{N_1}\mathrm{tr}\left(\mathbb{X}(\tau)(\mathbb{X}'(\tau)\mathbb{X}(\tau))^{-1}X'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}(\mathcal{Q}_4(\tau)-\bar{\mathcal{Q}}_4(\tau))\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathcal{G}(\zeta)\right)\\
&= \frac{1}{N_1^2}\mathrm{tr}\left((\frac{1}{N_1}\mathbb{X}'(\tau)\mathbb{X}(\tau))^{-1}X'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}(\mathcal{Q}_4(\tau)-\bar{\mathcal{Q}}_4(\tau))\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathcal{G}(\zeta)\mathbb{X}(\tau)\right),
\end{aligned}$$

where $\mathcal{G}(\zeta) = \mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\Gamma\mathbf{G}^{-1'}(\zeta_0)\mathbf{G}'(\zeta)$. By Assumption 7, the elements of $\left(\frac{1}{N_1}\mathbb{X}'(\tau)\mathbb{X}(\tau)\right)^{-1}$ are uniformly bounded for large enough N, uniformly in $\tau\in\Delta$. By Lemma A.1 and A.2, $X'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}(\mathcal{Q}_4(\tau)-\bar{\mathcal{Q}}_4(\tau))\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathcal{G}(\zeta)$ is uniformly bounded in both row sum and column sum matrix norms, uniformly in $\zeta\in\Delta$. These results imply that $\frac{1}{N_1}\mathrm{tr}\left(\Gamma\mathbf{G}^{-1'}(\zeta_0)\mathbf{P}_3^e(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)$ converges to 0 as $N\longrightarrow\infty$, uniformly in $\zeta\in\Delta$.

C.6 Proof of Theorem 4.2

By the mean value theorem, $\sqrt{N_1}(\widehat{\omega}^\dagger - \omega_0) = -\left(\frac{1}{N_1}\frac{\partial S^\dagger(\overline{\omega})}{\partial \omega'}\right)^{-1}\frac{1}{\sqrt{N_1}}S^\dagger(\omega_0)$, where $\overline{\omega}$ is between $\widehat{\omega}^\dagger$ and ω_0 elementwise. Thus we need to prove:

(i) $\frac{1}{\sqrt{N_1}}S^\dagger(\omega_0) \xrightarrow{d} N\left(0, \lim_{N\to\infty}\Omega^\dagger(\omega_0)\right)$,

(ii) $\frac{1}{N_1}\left(\frac{\partial S^\dagger(\overline{\omega})}{\partial\omega'} - \frac{\partial S^\dagger(\omega_0)}{\partial\omega'}\right) = o_p(1)$,

(iii) $\frac{1}{N_1}\left(\frac{\partial S^\dagger(\omega_0)}{\partial\omega'} - \mathrm{E}\left(\frac{\partial S^\dagger(\omega_0)}{\partial\omega'}\right)\right) = o_p(1)$.

Proof of (i). Since the elements of $S^\dagger(\omega_0)$ are linear quadratic forms in ϵ, we can find an $(k+2)\times 1$ vector $a = (a_1^{'}, a_2, a_3)^{'}$ such that $a^{'}S^\dagger(\omega_0) = b^{'}\epsilon + \epsilon^{'}B\epsilon$, where $b^{'} = a_1^{'}\mathbb{X}^{'}(\tau_0) - a_2\phi^{'}\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}^{'}}(\mathbf{S}^{'}(\tau_0) - \overline{\mathbf{S}}^{'}(\tau_0))\mathbb{Q}_{\mathbb{C}}(\tau_0) - a_3\delta_0^{'}\mathbb{C}^{'}(\tau_0)(\mathcal{Q}_4(\tau_0) - \overline{\mathcal{Q}}_4(\tau_0))\mathbb{Q}_{\mathbb{C}}(\tau_0)$ and $B = -a_2(\mathbf{S}^{'}(\tau_0) - \overline{\mathbf{S}}^{'}(\tau_0))\mathbb{Q}_{\mathbb{C}}(\tau_0) - a_3(\mathcal{Q}_4(\tau_0) - \overline{\mathcal{Q}}_4(\tau_0))\mathbb{Q}_{\mathbb{C}}(\tau_0)$. Since b and B satisfy the conditions for the CLT in Kelejian and Prucha (2001) by Lemma A.1 and A.2(i), $\frac{1}{N_1}a^{'}S^\dagger(\omega_0)$ is asymptotically normal. Then, the Cramér-Wold device leads to (i).

Proof of (ii). Given the explicit expressions for the elements of the hessian matrix $H^\dagger(\omega)$ in the main paper, we note that $\dot{\overline{\mathbf{S}}}^{'}(\tau)$ and $\dot{\overline{\mathcal{Q}}}_4(\tau)$ are diagonal matrices with uniformly bounded elements. By Lemma A.3, we know that $\frac{1}{N_1}H^\dagger(\omega_0) = O_p(1)$, which implies $\frac{1}{N_1}H^\dagger(\overline{\omega}) = O_p(1)$. Then similar to the proof of Theorem 3.2(ii), we can prove $\frac{1}{N_1}\left(\frac{\partial S^\dagger(\overline{\omega})}{\partial\omega'} - \frac{\partial S^\dagger(\omega_0)}{\partial\omega'}\right) = o_p(1)$ using Lemma A.2, A.3 and A.4 and the reduced form of y, given by $y = \mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}(\phi + \mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)$.

Proof of (iii). Substituting the reduced form of y into $\frac{1}{N_1}\left(\frac{\partial S^\dagger(\omega_0)}{\partial\omega'} - \mathrm{E}\left(\frac{\partial S^\dagger(\omega_0)}{\partial\omega'}\right)\right)$, we know each element is a linear or quadratic function of ϵ. For example, for $H^\dagger_{\tau\tau}(\omega_0)$,

$$
\begin{aligned}
&\frac{1}{N_1}\left(H^\dagger_{\tau\tau}(\omega_0) - \mathrm{E}(H^\dagger_{\tau\tau}(\omega_0))\right) = -\frac{1}{N_1}(\epsilon^{'}\mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M}^{'}\mathbb{D}(\zeta_0)\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon - \mathrm{E}(\epsilon^{'}\mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{M}^{'}\mathbb{D}(\zeta_0)\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon))\\
&+\frac{1}{N_1}(\delta_0^{'}C^{'}(\mathbf{M}^{'}\overline{\mathcal{Q}}_4(\tau_0) + \overline{\mathcal{Q}}_{4\tau}(\tau_0) - \mathbb{D}(\zeta_0))\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon - \mathrm{E}(\delta_0^{'}C^{'}(\mathbf{M}^{'}\overline{\mathcal{Q}}_4(\tau_0) + \overline{\mathcal{Q}}_{4\tau}(\tau_0) - \mathbb{D}(\zeta_0))\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon))\\
&+\frac{1}{N_1}(\epsilon^{'}(\mathbf{M}^{'}\overline{\mathcal{Q}}_4(\tau_0) + \overline{\mathcal{Q}}_{4\tau}(\tau_0) - \mathbb{D}(\zeta_0))\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon - \mathrm{E}(\epsilon^{'}(\mathbf{M}^{'}\overline{\mathcal{Q}}_4(\tau_0) + \overline{\mathcal{Q}}_{4\tau}(\tau_0) - \mathbb{D}(\zeta_0))\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon))\\
&= o_p(1),
\end{aligned}
$$

by Lemma A.3. For the rest of the elements, the proof is similar to that of $H^\dagger_{\tau\tau}(\omega_0)$ and thus are omitted.

C.7 Proof of Theorem 4.3

From the generic detailed expression of $\Omega^\dagger(\omega_0)$ we know that the ζ elements, i.e., $\Omega^\dagger_{\alpha\alpha}(\omega)$, $\Omega^\dagger_{\alpha\tau}(\omega)$ and $\Omega^\dagger_{\tau\tau}(\omega)$ are quadratic in δ, which are of the form: $\delta^{'}\mathbb{C}^{'}(\tau)\Xi_a^{'}(\tau)\Gamma\Xi_b(\tau)\mathbb{C}(\tau)\delta$, for $\Xi_a(\tau), \Xi_b(\tau) = \overline{\overline{\mathbf{S}}}(\tau)$ or $\overline{\overline{\mathcal{Q}}}(\tau)$. Similar to the homoskedastic case, we can apply the mean value theorem to $C\widehat{\delta}^\dagger$

with respect to the $\widehat{\tau}^\dagger$ element and get

$$\begin{aligned}C\widehat{\delta}^\dagger &= C\delta_0 + \mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau_0)\epsilon + \mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau_0)\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}\left((\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^\dagger\mathbf{W}} - \mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}})y - X(\widehat{\beta}^\dagger - \beta_0)\right)\\ &\quad + \xi(\bar{\tau})(\mathbf{e}^{\widehat{\boldsymbol{\alpha}}^\dagger\mathbf{W}}y - X\widehat{\beta}^\dagger)(\widehat{\tau}^\dagger - \tau_0),\end{aligned}$$

where $\bar{\tau}$ lies between $\widehat{\tau}^\dagger$ and τ_0 and $\xi(\bar{\tau}) = \frac{\partial \mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}}\mathbb{P}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}}{\partial\tau}|_{\tau=\bar{\tau}}$ are the same as those in the proof of Theorem 3.3. Substituting $C\widehat{\delta}^\dagger$ into the quadratic terms, we have

$$\begin{aligned}&\frac{1}{N_1}\widehat{\delta}^{\dagger'}\mathbb{C}'(\widehat{\tau}^\dagger)\Xi_a'(\widehat{\tau}^\dagger)\Gamma\Xi_b(\widehat{\tau}^\dagger)\mathbb{C}(\widehat{\tau}^\dagger)\widehat{\delta}^\dagger\\ &= \frac{1}{N_1}\delta_0'\mathbb{C}'(\tau_0)\Xi_a'(\tau_0)\Gamma\Xi_b(\tau_0)\mathbb{C}(\tau_0)\delta_0 + \frac{1}{N_1}\epsilon'\mathbb{P}_{\mathbb{C}}(\tau_0)\Xi_a'(\tau_0)\Gamma\Xi_b(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)\epsilon + o_p(1)\\ &= \frac{1}{N_1}\delta_0'\mathbb{C}'(\tau_0)\Xi_a'(\tau_0)\Gamma\Xi_b(\tau_0)\mathbb{C}(\tau_0)\delta_0 + \frac{1}{N_1}\mathrm{tr}\left(\Gamma\mathbb{P}_{\mathbb{C}}(\tau_0)\Xi_a'(\tau_0)\Gamma\Xi_b(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)\right) + o_p(1).\end{aligned}$$

Thus the bias matrix $\mathrm{Bias}_\delta^\dagger(\tau_0,\Gamma)$ can be written as

$$\mathrm{Bias}_\delta^\dagger(\tau_0,\Gamma) = \begin{pmatrix}0 & 0 & 0\\ 0 & \mathrm{Bias}^\dagger_{\delta,\alpha\alpha}(\tau_0,\Gamma) & \mathrm{Bias}^\dagger_{\delta,\alpha\tau}(\tau_0,\Gamma)\\ 0 & \mathrm{Bias}^\dagger_{\delta,\tau\alpha}(\tau_0,\Gamma) & \mathrm{Bias}^\dagger_{\delta,\tau\tau}(\tau_0,\Gamma)\end{pmatrix},$$

where

$$\begin{aligned}&\mathrm{Bias}^\dagger_{\delta,\alpha\alpha}(\tau_0,\Gamma) = \frac{1}{N_1}\mathrm{tr}\left(\Gamma\mathbb{P}_{\mathbb{C}}(\tau_0)\overline{\overline{\mathbf{S}}}'(\tau_0)\Gamma\overline{\overline{\mathbf{S}}}(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)\right),\\ &\mathrm{Bias}^\dagger_{\delta,\alpha\tau}(\tau_0,\Gamma) = \mathrm{Bias}^\dagger_{\delta,\tau\alpha}(\tau_0,\Gamma) = \frac{1}{N_1}\mathrm{tr}\left(\Gamma\mathbb{P}_{\mathbb{C}}(\tau_0)\overline{\overline{\mathbf{S}}}'(\tau_0)\Gamma\overline{\overline{\mathcal{Q}}}(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)\right), \text{ and}\\ &\mathrm{Bias}^\dagger_{\delta,\tau\tau}(\tau_0,\Gamma) = \frac{1}{N_1}\mathrm{tr}\left(\Gamma\mathbb{P}_{\mathbb{C}}(\tau_0)\overline{\overline{\mathcal{Q}}}'(\tau_0)\Gamma\overline{\overline{\mathcal{Q}}}(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)\right).\end{aligned}$$

C.8 Proof of Theorem 4.4

Note $\tilde{\epsilon}(\zeta) = \mathbb{Q}_{\mathbb{C}}(\tau)\epsilon(\zeta) = \mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y - X\beta)$. Let $\tilde{\epsilon}_j$ and $\widehat{\epsilon}_j$ be the jth element of $\tilde{\epsilon} = \tilde{\epsilon}(\zeta_0)$ and $\widehat{\epsilon} = \tilde{\epsilon}(\widehat{\zeta}^\dagger)$ respectively. Similar to (C.7), by the mean value theorem, $\widehat{\epsilon}_j \equiv \tilde{\epsilon}_j(\widehat{\omega}^\dagger) = \tilde{\epsilon}_j + \nu_j'(\widehat{\omega}^\dagger - \omega_0) + o_p\left(\left\|\widehat{\omega}^\dagger - \omega_0\right\|\right)$. Then, in vector form,

$$\widehat{\epsilon} = \tilde{\epsilon} + V(\widehat{\omega} - \omega_0) + o_p\left(\left\|\widehat{\omega}^\dagger - \omega_0\right\|\right),$$

where $V = (\nu_1,\ldots,\nu_N)'$, with ν_j being the same as those defined below (C.7). Define $\dot{\Pi}(\tau) = \frac{\partial\Pi(\tau)}{\partial\tau} = -2\Pi(\tau)\left(\dot{\mathbb{Q}}_{\mathbb{C}}(\tau)\odot\mathbb{Q}_{\mathbb{C}}(\tau)\right)\Pi(\tau)$. Then we can easily see that $\left\|\dot{\Pi}(\tau)\right\|_1$ and $\left\|\dot{\Pi}(\tau)\right\|_\infty$ are bounded in a neighborhood of τ_0. Let $\Pi_{jh}(\tau_0)$ and $\dot{\Pi}_{jh}(\tau_0)$ be the (j,h)th element of $\Pi(\tau_0)$ and $\dot{\Pi}(\tau_0)$. By the mean value theorem, $\Pi_{jh}(\widehat{\tau}^\dagger) = \Pi_{jh}(\tau_0) + \dot{\Pi}_{jh}(\bar{\tau})(\widehat{\tau}^\dagger - \tau_0) = \Pi_{jh}(\tau_0) + \dot{\Pi}_{jh}(\tau_0)(\widehat{\tau}^\dagger -$

$\tau_0) + o_p(\left\|\widehat{\tau}^\dagger - \tau_0\right\|)$, where $\bar{\tau}$ lies between $\widehat{\tau}^\dagger$ and τ_0. In matrix form, it becomes

$$\Pi(\widehat{\tau}^\dagger) = \Pi(\tau_0) + \dot{\Pi}(\tau_0)(\widehat{\tau}^\dagger - \tau_0) + o_p\left(\left\|\widehat{\tau}^\dagger - \tau_0\right\|\right).$$

Let $\widehat{\Sigma} = (\widehat{\sigma}_1^2, \ldots, \widehat{\sigma}_N^2)' = \Pi(\widehat{\tau}^\dagger)(\widehat{\epsilon} \odot \widehat{\epsilon})$ and $\tilde{\Sigma} = \Pi(\tau_0)(\tilde{\epsilon} \odot \tilde{\epsilon})$. Note that the elements of $\tilde{\epsilon}$ are $O_p(1)$, the elements of $\Pi(\tau_0)$ and $\dot{\Pi}(\tau_0)$ are $O(1)$, the elements in the rows of V are $O_p(1)$, and $\widehat{\omega}^\dagger - \omega_0 = O_p(1/\sqrt{N_1})$. Then, by MVT, we have:

$$\widehat{\Sigma} = \tilde{\Sigma} + 2\Pi(\tau_0)\left(\tilde{\epsilon} \odot V(\widehat{\omega}^\dagger - \omega_0)\right) + \dot{\Pi}(\tau_0)(\tilde{\epsilon} \odot \tilde{\epsilon})(\widehat{\tau}^\dagger - \tau_0) + o_p\left(\left\|\widehat{\tau}^\dagger - \tau_0\right\|\right). \tag{C.14}$$

Proof of (i)**.** Let $c = (c_{11}, \ldots, c_{NN})'$ be the $N \times 1$ vector containing the diagonal element of matrix C. Let $\Sigma = \mathrm{E}(\epsilon \odot \epsilon) = (\sigma_1^2, \ldots, \sigma_N^2)'$. Then

$$\frac{1}{N}\left(\mathrm{tr}(\widehat{\Gamma}C) - \mathrm{tr}(\Gamma C)\right) = \frac{1}{N}c'(\widehat{\Sigma} - \Sigma) = \frac{1}{N}c'(\widehat{\Sigma} - \tilde{\Sigma}) + \frac{1}{N}c'(\tilde{\Sigma} - \Sigma). \tag{C.15}$$

So we need to prove that the two terms in (C.15) are $o_p(1)$. For the first term, by (C.14),

$$\begin{aligned}
\frac{1}{N}c'(\widehat{\Sigma} - \tilde{\Sigma}) &= \frac{2}{N}c'\left(\Pi(\tau_0)(\tilde{\epsilon} \odot V(\widehat{\omega}^\dagger - \omega_0))\right) + \frac{1}{N}c'\dot{\Pi}(\tau_0)(\tilde{\epsilon} \odot \tilde{\epsilon})(\widehat{\tau}^\dagger - \tau_0) + o_p\left(\left\|\widehat{\tau}^\dagger - \tau_0\right\|\right) \\
&= \frac{2}{N}\sum_{j=1}^N c_{jj}\left(\sum_{h=1}^N \Pi_{jh}(\tau_0)\tilde{\epsilon}_h \nu_j'\right)(\widehat{\omega}^\dagger - \omega_0) + \frac{1}{N}\sum_{j=1}^N c_{jj}\left(\sum_{h=1}^N \dot{\Pi}_{jh}(\tau_0)\sum_{k=1}^N q_{hk}^2\sigma_k^2\right)(\widehat{\tau}^\dagger - \tau_0) \\
&\quad + o_p\left(\left\|\widehat{\tau}^\dagger - \tau_0\right\|\right) = o_p(1),
\end{aligned}$$

where q_{hk} is defined as the (h,k)th element of $\mathbb{Q}_{\mathbb{C}}(\tau_0)$. Here, the last equality holds since c_{jj} are uniformly bounded, $\Pi_{jh}(\tau_0)$ and $\dot{\Pi}_{jh}(\tau_0)$ are $O(1)$, $\tilde{\epsilon}_h$ are $O_p(1)$ and $\widehat{\omega}^\dagger - \omega_0 = o_p(1)$ for $j, h = 1, \ldots, N$. For the second term of (C.15), we have

$$\begin{aligned}
\tilde{\Sigma} &= \Pi(\tau_0)(\tilde{\epsilon} \odot \tilde{\epsilon}) = \Pi(\tau_0)\left((\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon) \odot (\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon)\right) \\
&= \Pi(\tau_0)\left((\mathbb{Q}_{\mathbb{C}}(\tau_0) \odot \mathbb{Q}_{\mathbb{C}}(\tau_0))(\epsilon \odot \epsilon) + \psi\right) = \epsilon \odot \epsilon + \Pi(\tau_0)\psi,
\end{aligned} \tag{C.16}$$

where ψ is an $N \times 1$ vector with jth element $\psi_j = \sum_{k=1}^N \epsilon_k h_{jk}$, where $h_{jk} = 2q_{jk}\sum_{l=1}^{k-1} q_{jl}\epsilon_l$, $k \geq 2$ and $h_{jl} = 0$. In the third equality of (C.16) we break $(\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon) \odot (\mathbb{Q}_{\mathbb{C}}(\tau_0)\epsilon) = (\sum_{k=1}^N q_{jk}\epsilon_k)^2$ into the sum of $(\mathbb{Q}_{\mathbb{C}}(\tau_0) \odot \mathbb{Q}_{\mathbb{C}}(\tau_0))(\epsilon \odot \epsilon) = \sum_{k=1}^N q_{jk}^2\epsilon_k^2$ and cross-multiplications ψ. Since ψ_j is $(\epsilon_1, \ldots, \epsilon_N)$ measurable, $\{\epsilon_k h_{jk}\}$ form an M.D. sequence. Thus

$$\frac{1}{N}c'(\tilde{\Sigma} - \Sigma) = \frac{1}{N}c'(\epsilon \odot \epsilon - \Sigma) + \frac{1}{N}c'\Pi(\tau_0)\psi = o_p(1), \tag{C.17}$$

where the first term is $o_p(1)$ by Lemma A.3(iv) and the second term is $o_p(1)$ by Theorem 19.7 for WLLN in Davidson (1994).

Proof of (ii)**.** Note that $\mathrm{tr}(\Gamma A \Gamma B) = \Sigma^{'}(A \odot B^{'})\Sigma$. Then

$$\frac{1}{N}\mathrm{tr}(\widehat{\Gamma} A \widehat{\Gamma} B) - \frac{1}{N}\mathrm{tr}(\Gamma A \Gamma B) = \frac{1}{N}\widehat{\Sigma}^{'}(A \odot B^{'})\widehat{\Sigma} - \frac{1}{N}\Sigma^{'}(A \odot B^{'})\Sigma$$
$$= \frac{1}{N}(\widehat{\Sigma}^{'}(A \odot B^{'})\widehat{\Sigma} - \tilde{\Sigma}^{'}(A \odot B^{'})\tilde{\Sigma}) + \frac{1}{N}(\tilde{\Sigma}^{'}(A \odot B^{'})\tilde{\Sigma} - \Sigma^{'}(A \odot B^{'})\Sigma). \tag{C.18}$$

For the first term in (C.18), note $\frac{1}{N}[\widehat{\Sigma}^{'}(A \odot B^{'})\widehat{\Sigma} - \tilde{\Sigma}^{'}(A \odot B^{'})\tilde{\Sigma}] = G_1 + G_2 + G_3$, where $G_1 = \frac{1}{N}(\widehat{\Sigma} - \tilde{\Sigma})^{'}(A \odot B^{'})(\widehat{\Sigma} - \tilde{\Sigma})$, $G_2 = \frac{1}{N}(\widehat{\Sigma} - \tilde{\Sigma})^{'}(A \odot B^{'})\tilde{\Sigma}$ and $G_3 = \frac{1}{N}\tilde{\Sigma}^{'}(A \odot B^{'})(\widehat{\Sigma} - \tilde{\Sigma})$. By the assumption of this theorem, A and B are uniformly bounded, thus $A \odot B$ is also uniformly bounded. Since $\tilde{\epsilon} = O_p(1)$, $V = O_p(1)$ and $\widehat{\omega}^{\dagger} - \omega_0 = O_p(1/\sqrt{N_1})$, by (C.14), $G_r = o_p(1)$ for $r = 1, 2, 3$. The proof is similar to that of $\frac{1}{N}c^{'}(\widehat{\Sigma} - \tilde{\Sigma}) = o_p(1)$ in the proof of (i) and thus is omitted. Therefore, the first term is $o_p(1)$.

For the second term in (C.18), we have $\frac{1}{N}[\tilde{\Sigma}^{'}(A \odot B^{'})\tilde{\Sigma} - \Sigma^{'}(A \odot B^{'})\Sigma] = G_4 + G_5 + G_6$, where $G_4 = \frac{1}{N}(\tilde{\Sigma} - \Sigma)^{'}(A \odot B^{'})(\tilde{\Sigma} - \Sigma)$, $G_5 = \frac{1}{N}(\tilde{\Sigma} - \Sigma)^{'}(A \odot B^{'})\Sigma$ and $G_6 = \frac{1}{N}\Sigma^{'}(A \odot B^{'})(\tilde{\Sigma} - \Sigma)$. For G_5, by (C.16), we have

$$G_5 = \frac{1}{N}(\epsilon \odot \epsilon - \Sigma)^{'}(A \odot B^{'})\Sigma + \frac{1}{N}\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Sigma = o_p(1),$$

by Lemma A.3(iv) and Theorem 19.7 for WLLN in Davidson (1994). Similarly, $G_6 = o_p(1)$.

For G_4, again by (C.16) we can write it as

$$\begin{aligned} G_4 &= \frac{1}{N}\psi^{'}\Pi(\tau_0)(A \odot B^{'})(\epsilon \odot \epsilon - \Sigma) + \frac{1}{N}(\epsilon \odot \epsilon - \Sigma)^{'}(A \odot B^{'})\Pi(\tau_0)\psi \\ &\quad + \frac{1}{N}(\epsilon \odot \epsilon - \Sigma)^{'}(A \odot B^{'})(\epsilon \odot \epsilon - \Sigma) + \frac{1}{N}\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)\psi \\ &= G_{4a} + G_{4b} + G_{4c} + G_{4d}. \end{aligned} \tag{C.19}$$

Consider G_{4a}. For simplicity, let us denote $S = \Pi(A \odot B^{'})$ with elements $\{s_{jk}\}$. Then,

$$\begin{aligned} G_{4a} &= \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N} s_{jk}\psi_j(\epsilon_k^2 - \sigma_k^2) \\ &= \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{\substack{m=1 \\ m\neq l}}^{N} s_{jk}q_{jl}q_{jm}(\epsilon_k^2 - \sigma_k^2)\epsilon_l\epsilon_m \\ &= \frac{1}{N}\sum_{k=1}^{N}[(\epsilon_k^2 - \sigma_k^2)\sum_{j=1}^{N}\sum_{l=1}^{k-1}\sum_{\substack{m=1 \\ m\neq l}}^{k-1} s_{jk}q_{jl}q_{jm}\epsilon_l\epsilon_m] + \frac{2}{N}\sum_{l=1}^{N}[\epsilon_l\sum_{j=1}^{N}\sum_{k=1}^{l-1}\sum_{\substack{m=1 \\ m\neq l}}^{l-1} s_{jk}q_{jl}q_{jm}\epsilon_m(\epsilon_k^2 - \sigma_k^2)] \\ &\quad + \frac{2}{N}\sum_{m=1}^{N}[\epsilon_m\sum_{j=1}^{N}\sum_{k=1}^{m-1} s_{jk}q_{jk}q_{jm}(\epsilon_k^3 - \mathrm{E}(\epsilon_k^3))] + \frac{2}{N}\sum_{m=1}^{N}[\epsilon_m\sum_{j=1}^{N}\sum_{\substack{k=1 \\ k\neq m}}^{N} s_{jk}q_{jk}q_{jm}(\mathrm{E}(\epsilon_k^3) - \epsilon_k^3)], \end{aligned} \tag{C.20}$$

which is the average of M.D. sequence. By Theorem 19.7 in Davidson (1994), (C.20) is $o_p(1)$. Similarly, G_{4b} is also $o_p(1)$.

For G_{4c}, recall $\Sigma = \mathrm{E}(\epsilon \odot \epsilon)$. Then $\mathrm{E}(G_{4c}) = \frac{1}{N}\mathrm{tr}((A \odot B^{'})\mathrm{Var}(\epsilon \odot \epsilon)) = 0$ because $\mathrm{Var}(\epsilon \odot \epsilon)$ is a diagonal matrix and $A \odot B^{'}$ has zero diagonals. By Lemma A.3(iv), $G_{4c} = \frac{1}{N}(\epsilon \odot \epsilon - \Sigma)^{'}(A \odot B^{'})(\epsilon \odot \epsilon - \Sigma) = o_p(1)$.

For the last term in (C.19), note each element of ψ is a sum of M.D. sequence as shown in the proof of (i). Also note that $\mathbb{Q}_{\mathbb{C}}(\tau_0)$ is a symmetric matrix, which implies $q_{ij} = q_{ji}$ for $i, j = 1, \dots, N$. Utilizing these facts, we can derive the following equation:

$$\mathrm{E}(\psi\psi^{'}) = 2(\mathbb{Q}_{\mathbb{C}}(\tau_0)\Gamma\mathbb{Q}_{\mathbb{C}}(\tau_0)) \odot (\mathbb{Q}_{\mathbb{C}}(\tau_0)\Gamma\mathbb{Q}_{\mathbb{C}}(\tau_0)) - 2(\mathbb{Q}_{\mathbb{C}}(\tau_0) \odot \mathbb{Q}_{\mathbb{C}}(\tau_0))\Gamma^2(\mathbb{Q}_{\mathbb{C}}(\tau_0) \odot \mathbb{Q}_{\mathbb{C}}(\tau_0)).$$

Thus,

$$\begin{aligned}\mathrm{E}\left(\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)\psi\right) &= 2\mathrm{tr}\left((A \odot B^{'})\Pi(\tau_0)\Lambda(\Gamma)\Pi(\tau_0)\right) - 2\mathrm{tr}\left((A \odot B)\Gamma^2\right) \\ &= 2\mathrm{tr}\left((A \odot B^{'})\Pi(\tau_0)\Lambda(\Gamma)\Pi(\tau_0)\right), \qquad \text{(C.21)}\end{aligned}$$

where $\Lambda(\Gamma) = (\mathbb{Q}_{\mathbb{C}}(\tau_0)\Gamma\mathbb{Q}_{\mathbb{C}}(\tau_0)) \odot (\mathbb{Q}_{\mathbb{C}}(\tau_0)\Gamma\mathbb{Q}_{\mathbb{C}}(\tau_0))$, and the second term equals zero because $A \odot B^{'}$ has zero diagonals and Γ^2 is a diagonal matrix.

Now let $\mathbb{Z} = \Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)$ with elements $\{z_{jk}\}$. It can be easily shown that $\{z_{jk}\}$ are uniformly bounded. Let $|z_{lm}| \leq \bar{z} < \infty$. Then

$$\begin{aligned}&\mathrm{Var}(\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)\psi) \\ &= 8\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{h=1}^{N}\sum_{\substack{p=1\\p\neq h}}^{N}\sum_{s=1}^{N}\sum_{\substack{r=1\\r\neq s}}^{N} z_{jk}z_{lm}q_{jh}q_{jp}q_{lh}q_{lp}q_{ks}q_{kr}q_{ms}q_{mr}\mathrm{E}(\epsilon_h^2\epsilon_p^2\epsilon_s^2\epsilon_r^2) \\ &\leq 8\bar{q}^2\bar{z}c\sum_{m=1}^{N}(\sum_{j=1}^{N}|s_{jk}|)(\sum_{j=1}^{N}|q_{kr}|)(\sum_{j=1}^{N}|q_{lp}|)(\sum_{j=1}^{N}|q_{lh}|)(\sum_{j=1}^{N}|q_{jp}|)(\sum_{j=1}^{N}|q_{ms}|)(\sum_{j=1}^{N}|q_{mr}|) \\ &= O(N),\end{aligned}$$

since $\mathrm{E}(\epsilon_h^2\epsilon_p^2\epsilon_s^2\epsilon_r^2)$ is equal to $\mathrm{E}(\epsilon_h^2\epsilon_s^2)\mathrm{E}(\epsilon_p^2\epsilon_r^2)$ or $\mathrm{E}(\epsilon_h^2\epsilon_r^2)\mathrm{E}(\epsilon_p^2\epsilon_s^2)$ due to the fact that $h \neq p$ and $s \neq r$, and one of them is less than a finite constant c. Then, by Chebyshev's inequality,

$$\begin{aligned}&P\left(\frac{1}{N}\left|\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)\psi - \mathrm{E}(\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)\psi)\right| \geq M\right) \\ &\leq \frac{1}{M^2}\frac{1}{N^2}\mathrm{Var}\left(\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)\psi\right) = o(1).\end{aligned}$$

It follows that $\frac{1}{N}\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)\psi - \frac{1}{N}\mathrm{E}\left(\psi^{'}\Pi(\tau_0)(A \odot B^{'})\Pi(\tau_0)\psi\right) = o_p(1)$. Thus $G_4 =$

$\frac{2}{N}\text{tr}\left((A \odot B')\Pi(\tau_0)\Lambda(\Gamma)\Pi(\tau_0)\right) + o_p(1)$. Combining the results for G_1 to G_6, we have

$$\frac{1}{N}\text{tr}(\widehat{\Gamma}A\widehat{\Gamma}B) - \frac{1}{N}\text{tr}(\Gamma A\Gamma B) = \sum_{r=1}^{6} G_r = \frac{2}{N}\text{tr}\left((A \odot B)\Pi(\tau_0)\Lambda(\Gamma)\Pi(\tau_0)\right) + o_p(1),$$

which completes the proof.

D Proof of Corollary 4.1

The proof for the consistency of $\widehat{\Psi}^{\dagger}(\widehat{\omega}^{\dagger})$ can be proved similar to part *(ii)* and *(iii)* in the proof of Theorem 4.2 and thus is omitted. For the consistency of $\widehat{\Omega}^{\dagger}$, note the following:

(1) The bias incurred by differences involving only ω, e.g., $\Omega^{\dagger}(\widehat{\omega}^{\dagger}, \delta_0, \Gamma) - \Omega^{\dagger}(\omega_0, \delta_0, \Gamma)$ and $\text{Bias}_{\Gamma}(\widehat{\tau}^{\dagger}, \Gamma) - \text{Bias}_{\Gamma}(\tau_0, \Gamma)$ disappears asymptotically since $\widehat{\omega}^{\dagger}$ is consistent;

(2) The asymptotic bias incurred by $\widehat{\Omega}^{\dagger}(\widehat{\omega}^{\dagger}, \widehat{\delta}^{\dagger}, \Gamma) - \widehat{\Omega}^{\dagger}(\widehat{\omega}^{\dagger}, \delta_0, \Gamma)$ is captured by $\text{Bias}_{\delta}^{\dagger}(\widehat{\tau}^{\dagger}, \Gamma)$;

(3) The asymptotic bias incurred by $\widehat{\Omega}^{\dagger}(\widehat{\omega}^{\dagger}, \widehat{\delta}^{\dagger}, \widehat{\Gamma}) - \widehat{\Omega}^{\dagger}(\widehat{\omega}^{\dagger}, \widehat{\delta}^{\dagger}, \Gamma)$ is captured by $\frac{2}{N_1}\text{tr}[(\Xi_a(\tau_0) \odot \Xi_b(\tau_0)^s)\Pi(\tau_0)\Lambda(\Gamma)\Pi(\tau_0)]$ for $a, b = \alpha, \tau$ and $\Xi_a(\tau_0), \Xi_b(\tau_0) = \bar{\bar{\mathbf{S}}}(\tau_0), \bar{\bar{\mathcal{Q}}}(\tau_0)$;

(4) What is left is to prove that the asymptotic bias incurred by $\text{Bias}_{\delta}^{\dagger}(\widehat{\tau}^{\dagger}, \widehat{\Gamma}) - \text{Bias}_{\delta}^{\dagger}(\widehat{\tau}^{\dagger}, \Gamma)$ is captured by $-\frac{2}{N_1}\text{tr}[(\mathbb{P}_{\mathbb{C}}(\tau_0)\Xi_a'(\tau_0) \odot \Xi_b(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0))\Pi(\tau_0)\Lambda(\Gamma)\Pi(\tau_0)]$.

Note $\text{Bias}_{\delta}^{\dagger}(\tau_0, \Gamma)$ has non-zero entries $\frac{1}{N_1}\text{tr}[\Gamma\mathbb{P}_{\mathbb{C}}(\tau_0)\Xi_a'(\tau_0)\Gamma\Xi_b(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)]$, for $a, b = \alpha, \tau$ and $\Xi_a(\tau_0), \Xi_b(\tau_0) = \bar{\bar{\mathbf{S}}}(\tau_0), \bar{\bar{\mathcal{Q}}}(\tau_0)$. Then we have

$$\begin{aligned}
&\frac{1}{N_1}\text{tr}[\mathbb{P}_{\mathbb{C}}(\widehat{\tau}^{\dagger})\Xi_a'(\widehat{\tau}^{\dagger})\widehat{\Gamma}\Xi_b(\widehat{\tau}^{\dagger})\mathbb{P}_{\mathbb{C}}(\widehat{\tau}^{\dagger})\widehat{\Gamma} - \mathbb{P}_{\mathbb{C}}(\tau_0)\Xi_a'(\tau_0)\Gamma\Xi_b(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)\Gamma] \\
&= \frac{1}{N_1}\text{tr}[\mathbb{P}_{\mathbb{C}}(\tau_0)\Xi_a'(\tau_0)\widehat{\Gamma}\Xi_b(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)\widehat{\Gamma} - \mathbb{P}_{\mathbb{C}}(\tau_0)\Xi_a'(\tau_0)\Gamma\Xi_b(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0)\Gamma] + o_p(1) \\
&= \frac{2}{N_1}\text{tr}[(\mathbb{P}_{\mathbb{C}}(\tau_0)\Xi_a'(\tau_0) \odot \Xi_b(\tau_0)\mathbb{P}_{\mathbb{C}}(\tau_0))\Pi(\tau_0)\Lambda(\Gamma)\Pi(\tau_0)] + o_p(1),
\end{aligned}$$

where the first equality follows from the mean value theorem and the second equality follows from Theorem 4.4. Then, together with the previous results, we have $\widehat{\Psi}^{\dagger-1}(\widehat{\omega}^{\dagger})\widehat{\Omega}^{\dagger}\widehat{\Psi}^{\dagger-1}(\widehat{\omega}^{\dagger}) - \Psi^{\dagger-1}(\omega_0)\Omega^{\dagger}(\omega_0)\Psi^{\dagger-1}(\omega_0) \xrightarrow{p} 0$.

E Details of the Empirical Application

Table E.1 includes the list of countries, and Table E.2 provides the list of industries. The FDI data are obtained from the Bureau of Economic Analysis. The GDP is measured in 2015 constant dollars, and the gross fixed capital formation (investment) are taken from the World Bank's World Development Indicators. Following Leamer (1984) and Baltagi et al. (2007), we estimate a country's capital stock using the perpetual inventory method. First, we choose a far enough year (1991) from the initial year in the dataset (2008) and estimate $K_{1993} = 2\sum_{t=1991}^{1995} I_t$, where I_t is the investment in year t. Then, we apply a depreciation rate of $\delta = 7\%$ to compute the capital stock at year t by $K_t = (1-\delta)K_{t-1} + I_t$. We obtain the educational attainment and labor data from Barro and Lee (2013). Since their datasets are at the 5-year frequency, we use the linear interpolation method to fill the missing data in the rest of the years. The skilled and unskilled labor endowments are then respectively computed as the labor times the percentage and one minus the percentage of population with at least tertiary education. Finally, the investment profile index is extracted from the International Country Risk Guide. Table E.3 provides the descriptive statistics for our sample. The estimation results are given in Table 5 in the main text. The pseudo-R^2 measure reported in the table is computed by $R^2 = 1 - \widehat{\epsilon}'\widehat{\epsilon}/\left((y-\bar{y})'(y-\bar{y})\right)$, where $\widehat{\epsilon} = \mathbf{e}^{\widehat{\tau}\mathbf{M}}\left(\mathbf{e}^{\widehat{\alpha}\mathbf{W}}y - X\widehat{\beta} - C\widehat{\delta}\right)$ and $\bar{y}$ is the sample mean of y.

Table E.1: List of Countries

Argentina	Australia	Austria	Brazil	Canada	Chile
China	Colombia	Costa Rica	Czech Republic	Denmark	Dominican Republic
Ecuador	Egypt	Finland	France	Germany	Greece
Honduras	Hong Kong	Hungary	India	Indonesia	Ireland
Israel	Italy	Japan	Malaysia	Mexico	Netherlands
New Zealand	Nigeria	Norway	Panama	Peru	Philippines
Poland	Portugal	Russia	Singapore	South Africa	Spain
Sweden	Switzerland	Thailand	Turkey	United Kingdom	

Table E.2: List of Industries

Food	Chemicals	Primary and fabricated metals
Machinery	Computers and electronics	Electrical equipments
Transportation equipment	Other manufacturing	Wholesale trade
Finance and insurance		

Table E.3: Descriptive statistics of our sample data

Variable	Mean	Std.Dev.	Min	Max
LFDI (Log of outward FDI)	6.12	2.24	0.00	12.03
G (Bilateral country size)	30.51	0.09	30.40	30.96
S (Similarity)	−2.88	1.08	−6.16	−0.76
k (Relative capital stock)	3.42	1.32	0.40	6.96
h^s (Relative skilled labor endowment)	2.80	1.35	−0.09	7.05
h^u (Relative unskilled labor endowment)	2.30	1.49	−1.78	4.69
Ψ	104.19	40.06	12.24	211.67
Φ	10.02	10.36	−5.97	35.30
R (Risk)	9.55	1.97	3.04	12.00

F Details of the Identification Conditions

Assumptions 6 and 9 in the main text state the identification conditions for ζ_0 under the homoskedastic and heteroskedastic cases, respectively. However, they are high level assumptions. In this section we derive some low level conditions that are sufficient for the identification of ζ_0. We will present the derivation for the homoskedastic case only. For the heteroskedastic case, the derivation follows along the same lines.

Because the M-estimation approach is equivalent to the method of moments approach under the exact identification case, the identification of ζ_0 requires that for $\zeta \neq \zeta_0$ it must be the case that $\overline{S}^{c*}(\zeta) \neq 0$. Before we proceed with the derivation, let us restate some existing results. From equation (C.1), we have

$$\begin{aligned}\bar{\sigma}_{\epsilon}^{*2}(\zeta) &= \frac{1}{N_1}\phi' \mathbf{e}^{-\boldsymbol{\alpha}_0 \mathbf{W}'}\mathbf{G}'(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi \\ &\quad + \frac{\sigma_{\epsilon 0}^2}{N_1}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(\mathbf{G}'(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}'(\zeta)\right).\end{aligned}$$

Also recall that $y = \mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}(\phi + \mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)$, $\phi = X\beta_0 + C\delta_0$, $y = \mathrm{E}(y) + \mathbf{G}^{-1}(\zeta_0)\epsilon$, equation (3.13)

$$\begin{aligned}\bar{\epsilon}(\zeta) &= \mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\big(y - \mathrm{E}(y)\big) + \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y \\ &= \mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon + \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y\end{aligned}$$

and equation (3.14)

$$\overline{S}^{c*}(\zeta) = \begin{cases} \alpha: & -\frac{1}{\bar{\sigma}_{\epsilon}^{*2}(\zeta)}\mathrm{E}\left(y'\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\bar{\epsilon}(\zeta)\right) + \mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\mathbf{W}\mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}}\right), \\ \tau: & -\frac{1}{\bar{\sigma}_{\epsilon}^{*2}(\zeta)}\mathrm{E}\left(\bar{\epsilon}'(\zeta)\mathbf{M}\bar{\epsilon}(\zeta)\right) + \mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\right). \end{cases}$$

Consider the expectation term in the α component of $\overline{S}^{c*}(\zeta)$. Substituting the definition of $\bar{\epsilon}(\zeta)$

and then the definition of y, we obtain

$$
\begin{aligned}
&\mathrm{E}\left(y^{'}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\left(\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon+\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y\right)\right)\\
&=\mathrm{E}\left(y^{'}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon\right)+\mathrm{E}\left(y^{'}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y\right)\\
&=\mathrm{E}\left((\phi+\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon\right)\\
&\quad+\mathrm{E}\left((\phi+\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}(\phi+\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon)\right)\\
&=\mathrm{E}\left(\epsilon^{'}\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}^{'}}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon\right)\\
&\quad+\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\\
&\quad+\mathrm{E}\left(\epsilon^{'}\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}^{'}}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}}\epsilon\right)\\
&=\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\\
&\quad+\mathrm{E}\left(\epsilon^{'}\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}^{'}}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon\right)\\
&=\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\\
&\quad+\sigma_{\epsilon 0}^2\mathrm{tr}\left(\mathbf{e}^{-\boldsymbol{\tau}_0\mathbf{M}^{'}}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)\\
&=\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\\
&\quad+\sigma_{\epsilon 0}^2\mathrm{tr}\left(\mathbf{G}^{-1'}(\zeta_0)\mathbf{G}^{'}(\zeta)\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\right)\\
&=\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\\
&\quad+\sigma_{\epsilon 0}^2\mathrm{tr}\left(\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right)
\end{aligned}
$$

where $\mathbf{S}(\tau)=\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\mathbf{W}\mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}}$ and $\mathbf{G}(\zeta_0)=\mathbf{e}^{\boldsymbol{\tau}_0\mathbf{M}}\mathbf{e}^{\boldsymbol{\alpha}_0\mathbf{W}}$. Notice that the second term in the α component can be written as $\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\mathbf{W}\mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}}\right)=\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{S}(\tau)\right)=\mathrm{tr}\left(\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\right)$. Multiplying the first term with $-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}$ and adding the second term, the α component becomes

$$
\begin{aligned}
&-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\mathrm{E}\left(y^{'}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}^{'}}\mathbf{W}^{'}\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}^{'}}\bar{\epsilon}(\zeta)\right)+\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\mathbf{W}\mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}}\right)\\
&\quad=-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\left(\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\right)\\
&\qquad-\frac{\sigma_{\epsilon 0}^2}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\mathrm{tr}\left(\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right)+\mathrm{tr}\left(\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\right)\\
&\quad=-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\left(\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\right)\\
&\qquad+\mathrm{tr}\left(\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\left(I_N-\frac{\sigma_{\epsilon 0}^2}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right)\right).
\end{aligned}
$$

Hence, the identification of ζ_0 follows, if for $\zeta \neq \zeta_0$, we have

$$\begin{aligned} & -\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\left(\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\right) \\ & \qquad + \operatorname{tr}\left(\mathbf{S}^{'}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\left(I_N - \frac{\sigma_{\epsilon 0}^2}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right)\right) \neq 0. \end{aligned} \tag{F.1}$$

Notice that when $\zeta = \zeta_0$ (which implies $\overline{\sigma}_\epsilon^{*2}(\zeta_0) = \sigma_{\epsilon 0}^2$), we have $\mathbb{Q}_{\mathbb{X}}(\tau_0)\mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{G}(\zeta_0)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi = \mathbf{0}$ and $I_N - \frac{\sigma_{\epsilon 0}^2}{\overline{\sigma}_\epsilon^{*2}(\zeta_0)}\mathbf{G}(\zeta_0)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta_0) = \mathbf{0}$.

Next, consider the expectation term in the τ component of $\overline{S}^{c*}(\zeta)$. Substituting the definition of $\bar{\epsilon}(\zeta)$ and then the definition of y, we obtain

$$\begin{aligned} \mathrm{E}\left(\bar{\epsilon}^{'}(\zeta)\mathbf{M}\bar{\epsilon}(\zeta)\right) &= \mathrm{E}\Big(\left(\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon + \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y\right)^{'}\mathbf{M} \\ & \qquad \times \left(\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon + \mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y\right)\Big) \\ &= \mathrm{E}\left(\epsilon^{'}\mathbf{G}^{-1'}(\zeta_0)\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{P}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon\right) \\ & \quad + \mathrm{E}\left(\epsilon^{'}\mathbf{G}^{-1'}(\zeta_0)\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{P}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y\right) \\ & \quad + \mathrm{E}\left(y^{'}\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{G}^{-1}(\zeta_0)\epsilon\right) \\ & \quad + \mathrm{E}\left(y^{'}\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)y\right) \\ &= \sigma_{\epsilon 0}^2\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{P}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{P}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right) \\ & \quad + 2\sigma_{\epsilon 0}^2\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{P}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right) \\ & \quad + \sigma_{\epsilon 0}^2\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right) \\ & \quad + \phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi \\ &= \sigma_{\epsilon 0}^2\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right) \\ & \quad + \phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi. \end{aligned}$$

Multiplying the first term with $-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}$ and adding the second term, and noting that $\operatorname{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\right) =$

$\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{C}}(\tau)\right)$, the τ component can be written as

$$
\begin{aligned}
&-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\mathrm{E}\left(\bar{\epsilon}^{'}(\zeta)\mathbf{M}\bar{\epsilon}(\zeta)\right)+\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\right)\\
&\quad=-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\left(\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\right)\\
&\qquad-\frac{\sigma_{\epsilon 0}^2}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right)+\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{C}}(\tau)\right)\\
&\quad=-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\left(\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\right)\\
&\qquad+\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{C}}(\tau)\left(I_N-\frac{\sigma_{\epsilon 0}^2}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right)\right).
\end{aligned}
$$

Hence, the identification of ζ_0 follows, if for $\zeta\neq\zeta_0$,

$$
\begin{aligned}
&-\frac{1}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\left(\phi^{'}\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}^{'}}\mathbf{G}^{'}(\zeta)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbb{Q}_{\mathbb{X}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{X}}(\tau)\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{G}(\zeta)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi\right)\\
&\quad+\mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\mathbb{Q}_{\mathbb{C}}(\tau)\left(I_N-\frac{\sigma_{\epsilon 0}^2}{\overline{\sigma}_\epsilon^{*2}(\zeta)}\mathbf{G}(\zeta)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta)\right)\right)\neq 0.
\end{aligned}
\tag{F.2}
$$

Note again that when $\zeta=\zeta_0$ (which implies $\overline{\sigma}_\epsilon^{*2}(\zeta_0)=\sigma_{\epsilon 0}^2$), we have $\mathbb{Q}_{\mathbb{X}}(\tau_0)\mathbb{Q}_{\mathbb{C}}(\tau_0)\mathbf{G}(\zeta_0)\mathbf{e}^{-\boldsymbol{\alpha}_0\mathbf{W}}\phi=\mathbf{0}$ and $I_N-\frac{\sigma_{\epsilon 0}^2}{\overline{\sigma}_\epsilon^{*2}(\zeta_0)}\mathbf{G}(\zeta_0)\left(\mathbf{G}^{'}(\zeta_0)\mathbf{G}(\zeta_0)\right)^{-1}\mathbf{G}^{'}(\zeta_0)=\mathbf{0}$.

In light of these results, the identification of of ζ_0 follows if either (F.1) or (F.2) holds for $\zeta\neq\zeta_0$.

G Pseudo Estimation Algorithms

Algorithm 1. M-estimation in the homoskedastic case

Require: y, X, W, M, C, n, T

$N \leftarrow length(y)$

Ensure: C is $N \times (n+T-1)$

$\widehat{\zeta}^* \leftarrow \texttt{SOLVE}(S^{c*}(\zeta)=0)$, where $S^{c*}(\zeta)$ is given in equation (3.10) below:

$$S^{c*}(\zeta) = \begin{cases} \alpha: & -\frac{1}{\widehat{\sigma}_\epsilon^{*2}(\zeta)} y' \mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}'}\mathbf{W}'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\widehat{\epsilon}(\zeta) + \mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\mathbf{W}\mathbf{e}^{-\boldsymbol{\tau}\mathbf{M}}\right), \\ \tau: & -\frac{1}{\widehat{\sigma}_\epsilon^{*2}(\zeta)} \widehat{\epsilon}'(\zeta)\mathbf{M}\widehat{\epsilon}(\zeta) + \mathrm{tr}\left(\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{M}\right). \end{cases}$$

Calculate: $\widehat{\beta}^* = \widehat{\beta}^*(\widehat{\zeta}^*)$ using $\widehat{\zeta}^*$ and equation (3.8) below:

$$\widehat{\beta}^*(\zeta) = \left(X'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}X\right)^{-1} X'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y.$$

Calculate: $\widehat{\sigma}_\epsilon^* = \widehat{\sigma}_\epsilon^*(\widehat{\zeta}^*)$ using $\widehat{\zeta}^*$ and equation (3.9) below:

$$\widehat{\sigma}_\epsilon^{*2}(\zeta) = \widehat{\epsilon}'(\zeta)\widehat{\epsilon}(\zeta)/N_1,$$

where $\widehat{\epsilon}(\zeta) = \tilde{\epsilon}(\widehat{\beta}^*(\zeta), \zeta)$.

Calculate: $\Psi^*(\widehat{\theta}^*)$, where $\widehat{\theta}^* = (\widehat{\beta}^{*\prime}, \widehat{\sigma}_\epsilon^{*2}, \widehat{\zeta}^{*\prime})'$.

Calculate: $\widehat{\rho}$ and $\widehat{\kappa}$ using equations (3.21) and (3.24) below:

$$\widehat{\rho} = \frac{\sum_{j=1}^N \widehat{\epsilon}_j^3}{\widehat{\sigma}_\epsilon^{*3}\sum_{j=1}^N\sum_{h=1}^N \widehat{q}_{jh}^3},$$

$$\widehat{\kappa} = \frac{\sum_{j=1}^N \widehat{\epsilon}_j^4 - 3\widehat{\sigma}_\epsilon^{*4}\sum_{j=1}^N\sum_{h=1}^N\sum_{l=1}^N \widehat{q}_{jh}^2\widehat{q}_{jl}^2}{\widehat{\sigma}_\epsilon^{*4}\sum_{j=1}^N\sum_{h=1}^N \widehat{q}_{jh}^4}.$$

Calculate: $\widehat{\Omega}^* = \Omega^*(\widehat{\theta}^*) - \mathrm{Bias}^*(\widehat{\tau}^*)$ using equation (3.19) and Theorem 3.3.

Calculate: $\Psi^{*-1}(\widehat{\theta}^*)\widehat{\Omega}^*\Psi^{*\prime-1}(\widehat{\theta}^*)$.

Algorithm 2. M-estimation in the heteroskedastic case

Require: y, X, W, M, C, n, T

$N \leftarrow length(y)$

Ensure: C is $N \times (n+T-1)$

$\widehat{\zeta}^\dagger \leftarrow \texttt{SOLVE}\left(S^{c\dagger}(\zeta)=0\right)$, where $S^{c\dagger}(\zeta)$ is given in equation (4.9) below:

$$S^{\dagger c}(\zeta) = \begin{cases} \alpha: & -y'\mathbf{G}'(\zeta)\left(\mathbf{S}'(\tau) - \overline{\mathbf{S}}'(\tau)\right)\widehat{\epsilon}(\zeta), \\ \tau: & -\left(\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y - X\widehat{\beta}^\dagger(\zeta)\right)'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\left(\mathcal{Q}_4(\tau) - \overline{\mathcal{Q}}_4(\tau)\right)\widehat{\epsilon}(\zeta). \end{cases}$$

Calculate: $\widehat{\beta}^\dagger = \widehat{\beta}^\dagger(\widehat{\zeta}^\dagger)$ using $\widehat{\zeta}^\dagger$ and equation (4.8) below:

$$\widehat{\beta}^\dagger(\zeta) = \left(X'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}X\right)^{-1} X'\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}'}\mathbb{Q}_{\mathbb{C}}(\tau)\mathbf{e}^{\boldsymbol{\tau}\mathbf{M}}\mathbf{e}^{\boldsymbol{\alpha}\mathbf{W}}y.$$

Calculate: $\Psi^\dagger(\widehat{\omega}^\dagger)$, where $\widehat{\omega}^\dagger = (\widehat{\beta}^{\dagger\prime}, \widehat{\zeta}^{\dagger\prime})'$.

Calculate: $\widehat{\Omega}^\dagger = \Omega^\dagger(\widehat{\omega}^\dagger, \widehat{\delta}^\dagger, \widehat{\Gamma}) - \mathrm{Bias}_\delta^\dagger(\widehat{\tau}^\dagger, \widehat{\Gamma}) - \mathrm{Bias}_\Gamma^\dagger(\widehat{\tau}^\dagger, \widehat{\Gamma})$ using equation (4.14), Theorem 4.3 and 4.4.

Calculate: $\Psi^{\dagger-1}(\widehat{\omega}^\dagger)\widehat{\Omega}^\dagger\Psi^{\dagger\prime-1}(\widehat{\omega}^\dagger)$.

H Additional Simulation Results

In this section, we provide the simulation results for two additional specifications for the variance terms: (i) $\sigma_{it}^2 = 1 - \kappa_1^2 + \kappa_2 \frac{i \times t}{nT}$, with $\kappa_1 = 0.8$ and $\kappa_2 = 1.5$, and (ii) $\sigma_{it}^2 = \frac{|X_{1,it}| + |X_{2,it}|}{\frac{1}{n}\sum_{i=1}^{n}(|X_{1,it}| + |X_{2,it}|)}$. The results based on these cases are similar to our main results based on $\sigma_{it}^2 = \exp(0.1 + 0.35 X_{2,it})$.

Table H.1: Heteroskedastic case with $\eta_{it} \sim N(0,1)$ and $\sigma_{it}^2 = 1 - \kappa_1^2 + \kappa_2 \frac{i \times t}{nT}$

	W=Rook, M=Queen			W=Queen, M=Rook		
	QMLE	ME	RME	QMLE	ME	RME
			$n=50$, $T=3$			
$\beta_{10}=-1$	−1.0009(.089)	−0.9998(.089)[.086]	−0.9998(.089)[.084]	−0.9992(.073)	−1.0005(.073)[.075]	−1.0007(.073)[.070]
$\beta_{20}=2$	1.9966(.085)	1.9954(.085)[.079]	1.9955(.085)[.078]	1.9985(.067)	1.9981(.067)[.067]	1.9982(.067)[.065]
$\alpha_0=-2$	−1.9949(.051)	−2.0001(.051)[.052]	−2.0002(.051)[.049]	−1.9926(.065)	−1.9988(.065)[.067]	−1.9992(.066)[.065]
$\tau_0=-1$	−0.9745(.185)	−1.0123(.181)[.184]	−1.0108(.181)[.182]	−1.0432(.167)	−1.0211(.152)[.160]	−1.0228(.152)[.156]
$\sigma_0^2=1$	0.5240(.089)	0.8240(.140)[.130]	–	0.5103(.084)	0.8088(.133)[.129]	–
			$n=50$, $T=7$			
$\beta_{10}=-1$	−0.9976(.050)	−0.9981(.050)[.048]	−0.9981(.050)[.048]	−0.9997(.043)	−0.9998(.043)[.041]	−0.9998(.043)[.041]
$\beta_{20}=2$	1.9993(.046)	1.9996(.046)[.045]	1.9996(.046)[.045]	1.9985(.040)	1.9994(.040)[.040]	1.9995(.040)[.041]
$\alpha_0=-2$	−1.9937(.032)	−1.9998(.032)[.033]	−2.0000(.032)[.033]	−1.9953(.037)	−2.0020(.037)[.036]	−2.0024(.037)[.037]
$\tau_0=-1$	−0.9534(.111)	−0.9985(.109)[.103]	−0.9989(.109)[.106]	−0.9947(.084)	−1.0096(.081)[.081]	−1.0108(.081)[.082]
$\sigma_0^2=1$	0.6522(.059)	0.7877(.072)[.076]	–	0.6488(.058)	0.7811(.069)[.074]	–
			n=100, $T=3$			
$\beta_{10}=-1$	−1.0037(.066)	−1.0045(.066)[.064]	−1.0045(.066)[.065]	−1.0001(.055)	−1.0001(.055)[.056]	−1.0001(.055)[.056]
$\beta_{20}=2$	1.9991(.059)	1.9986(.059)[.059]	1.9986(.059)[.058]	2.0013(.050)	2.0006(.050)[.050]	2.0005(.050)[.049]
$\alpha_0=-2$	−1.9963(.040)	−1.9991(.040)[.041]	−1.9991(.040)[.041]	−1.9925(.052)	−1.9968(.052)[.049]	−1.9979(.052)[.049]
$\tau_0=-1$	−1.0016(.134)	−0.9973(.130)[.130]	−1.0005(.130)[.132]	−1.0289(.104)	−1.0095(.099)[.101]	−1.0102(.099)[.100]
$\sigma_0^2=1$	0.5427(.063)	0.8572(.100)[.100]	–	0.5378(.061)	0.8362(.096)[.095]	–
			n=100, $T=7$			
$\beta_{10}=-1$	−1.0003(.033)	−1.0005(.033)[.033]	−1.0005(.033)[.033]	−0.9997(.029)	−0.9997(.029)[.029]	−0.9998(.029)[.029]
$\beta_{20}=2$	1.9985(.033)	1.9985(.033)[.033]	1.9985(.033)[.032]	2.0015(.031)	2.0010(.031)[.030]	2.0009(.031)[.030]
$\alpha_0=-2$	−1.9967(.022)	−1.9995(.022)[.022]	−1.9996(.022)[.022]	−1.9941(.027)	−1.9978(.027)[.026]	−1.9985(.027)[.026]
$\tau_0=-1$	−0.9811(.074)	−1.0012(.074)[.070]	−1.0015(.074)[.071]	−1.0111(.061)	−1.0059(.058)[.057]	−1.0055(.058)[.058]
$\sigma_0^2=1$	0.6603(.043)	0.7850(.051)[.053]	–	0.6578(.043)	0.7872(.052)[.054]	–
			$n=50$, $T=3$			
$\beta_{10}=-1$	−1.0029(.087)	−1.0026(.086)[.082]	−1.0029(.087)[.081]	−0.9967(.087)	−0.9964(.086)[.081]	−0.9962(.086)[.084]
$\beta_{20}=2$	2.0024(.089)	2.0008(.087)[.084]	2.0010(.087)[.082]	1.9965(.079)	1.9980(.078)[.080]	1.9979(.078)[.075]
$\alpha_0=-2$	−1.9927(.063)	−1.9985(.062)[.059]	−1.9988(.064)[.060]	−1.9911(.068)	−1.9959(.068)[.069]	−1.9965(.068)[.069]
$\tau_0=1$	1.2345(.262)	1.0777(.223)[.222]	1.0759(.228)[.223]	1.1182(.163)	1.0245(.151)[.156]	1.0296(.157)[.158]
$\sigma_0^2=1$	0.5273(.087)	0.8341(.138)[.133]	–	0.5125(.086)	0.8156(.137)[.132]	–
			$n=50$, $T=7$			
$\beta_{10}=-1$	−0.9977(.045)	−0.9977(.045)[.044]	−0.9978(.045)[.045]	−1.0015(.040)	−1.0009(.040)[.042]	−1.0009(.040)[.041]
$\beta_{20}=2$	2.0005(.041)	1.9994(.041)[.040]	1.9994(.041)[.039]	2.0018(.039)	2.0010(.039)[.040]	2.0009(.039)[.040]
$\alpha_0=-2$	−1.9962(.030)	−2.0014(.030)[.032]	−2.0018(.030)[.031]	−1.9913(.037)	−1.9972(.037)[.035]	−1.9978(.038)[.036]
$\tau_0=1$	1.1170(.126)	1.0232(.118)[.112]	1.0190(.117)[.113]	1.0599(.087)	1.0093(.085)[.081]	1.0094(.085)[.082]
$\sigma_0^2=1$	0.6481(.061)	0.7842(.073)[.075]	–	0.6509(.061)	0.7847(.073)[.075]	–
			n=100, $T=3$			
$\beta_{10}=-1$	−0.9991(.057)	−0.9985(.057)[.057]	−0.9987(.057)[.056]	−0.9997(.059)	−0.9994(.059)[.057]	−0.9994(.059)[.055]
$\beta_{20}=2$	1.9972(.061)	1.9977(.061)[.059]	1.9978(.060)[.057]	1.9977(.053)	1.9985(.053)[.053]	1.9985(.053)[.053]
$\alpha_0=-2$	−1.9976(.039)	−1.9998(.039)[.038]	−1.9996(.039)[.039]	−1.9894(.051)	−1.9942(.051)[.051]	−1.9943(.051)[.052]
$\tau_0=1$	1.1701(.176)	1.0246(.145)[.149]	1.0273(.147)[.152]	1.0694(.102)	1.0134(.096)[.100]	1.0135(.096)[.100]
$\sigma_0^2=1$	0.5332(.060)	0.8468(.094)[.099]	–	0.5442(.059)	0.8474(.092)[.097]	–
			n=100, $T=7$			
$\beta_{10}=-1$	−0.9980(.031)	−0.9981(.031)[.031]	−0.9981(.031)[.031]	−0.9992(.029)	−0.9995(.029)[.029]	−0.9995(.029)[.029]
$\beta_{20}=2$	1.9990(.030)	1.9993(.030)[.030]	1.9993(.030)[.030]	2.0002(.032)	1.9998(.032)[.030]	1.9998(.032)[.030]
$\alpha_0=-2$	−1.9974(.022)	−2.0000(.022)[.021]	−1.9999(.022)[.022]	−1.9954(.028)	−1.9992(.028)[.027]	−1.9995(.028)[.027]
$\tau_0=1$	1.0583(.075)	1.0128(.072)[.073]	1.0119(.073)[.074]	1.0438(.060)	1.0034(.058)[.057]	1.0038(.058)[.059]
$\sigma_0^2=1$	0.6594(.043)	0.7843(.052)[.053]	–	0.6614(.044)	0.7922(.053)[.054]	–

Notes: We report the empirical mean (standard deviation) [average asymptotic standard error].

Table H.2: Heteroskedastic case with $\eta_{it} \sim N(0,1)$ and $\sigma_{it}^2 = \frac{|X_{1,it}|+|X_{2,it}|}{\frac{1}{n}\sum_{i=1}^{n}(|X_{1,it}|+|X_{2,it}|)}$

	W=Rook, M=Queen			W=Queen, M=Rook		
	QMLE	ME	RME	QMLE	ME	RME
			$n = 50$, $T = 3$			
$\beta_{10} = -1$	−1.0078(.091)	−1.0064(.091)[.093]	−1.0064(.091)[.089]	−0.9954(.081)	−0.9966(.081)[.081]	−0.9969(.081)[.078]
$\beta_{20} = 2$	1.9964(.097)	1.9952(.097)[.084]	1.9954(.097)[.092]	1.9982(.085)	1.9977(.085)[.073]	1.9980(.085)[.078]
$\alpha_0 = -2$	−1.9937(.055)	−1.9994(.055)[.055]	−1.9989(.055)[.055]	−1.9880(.071)	−1.9949(.071)[.072]	−1.9947(.072)[.070]
$\tau_0 = -1$	−0.9640(.184)	−1.0017(.181)[.182]	−1.0041(.181)[.180]	−1.0473(.167)	−1.0252(.153)[.158]	−1.0306(.153)[.152]
$\sigma_0^2 = 1$	0.5986(.097)	0.9412(.153)[.150]	–	0.5974(.098)	0.9469(.155)[.150]	–
			$n = 50$, $T = 7$			
$\beta_{10} = -1$	−1.0008(.055)	−1.0014(.055)[.054]	−1.0014(.055)[.054]	−0.9983(.048)	−0.9984(.048)[.047]	−0.9984(.048)[.047]
$\beta_{20} = 2$	2.0031(.056)	2.0034(.056)[.050]	2.0034(.056)[.055]	1.9965(.048)	1.9977(.048)[.045]	1.9977(.048)[.048]
$\alpha_0 = -2$	−1.9941(.036)	−2.0013(.036)[.036]	−2.0015(.036)[.035]	−1.9893(.039)	−1.9972(.039)[.039]	−1.9971(.040)[.039]
$\tau_0 = -1$	−0.9587(.105)	−1.0040(.104)[.103]	−1.0040(.104)[.103]	−0.9959(.078)	−1.0106(.076)[.079]	−1.0102(.076)[.079]
$\sigma_0^2 = 1$	0.8109(.075)	0.9793(.090)[.094]	–	0.8227(.077)	0.9906(.093)[.093]	–
			$n = 100$, $T = 3$			
$\beta_{10} = -1$	−0.9951(.067)	−0.9960(.067)[.068]	−0.9961(.067)[.066]	−1.0015(.063)	−1.0015(.063)[.061]	−1.0016(.063)[.060]
$\beta_{20} = 2$	2.0011(.070)	2.0006(.070)[.062]	2.0005(.070)[.067]	2.0030(.054)	2.0023(.054)[.054]	2.0023(.054)[.054]
$\alpha_0 = -2$	−1.9956(.043)	−1.9988(.043)[.043]	−1.9989(.043)[.042]	−1.9948(.054)	−1.9995(.054)[.052]	−1.9998(.054)[.052]
$\tau_0 = -1$	−1.0050(.136)	−1.0007(.132)[.130]	−1.0017(.133)[.130]	−1.0293(.109)	−1.0101(.103)[.100]	−1.0129(.104)[.099]
$\sigma_0^2 = 1$	0.6166(.072)	0.9740(.114)[.113]	–	0.6326(.071)	0.9835(.111)[.108]	–
			$n = 100$, $T = 7$			
$\beta_{10} = -1$	−1.0019(.037)	−1.0020(.037)[.037]	−1.0020(.037)[.037]	−1.0003(.032)	−1.0003(.032)[.033]	−1.0003(.032)[.033]
$\beta_{20} = 2$	2.0005(.042)	2.0006(.042)[.037]	2.0006(.042)[.042]	2.0008(.037)	2.0003(.037)[.034]	2.0003(.037)[.037]
$\alpha_0 = -2$	−1.9979(.023)	−2.0012(.023)[.024]	−2.0012(.023)[.023]	−1.9951(.028)	−1.9994(.028)[.029]	−1.9995(.028)[.028]
$\tau_0 = -1$	−0.9805(.072)	−1.0006(.072)[.070]	−1.0006(.072)[.070]	−1.0103(.055)	−1.0051(.053)[.057]	−1.0041(.053)[.057]
$\sigma_0^2 = 1$	0.8326(.055)	0.9898(.065)[.066]	–	0.8305(.054)	0.9939(.065)[.068]	–
			$n = 50$, $T = 3$			
$\beta_{10} = -1$	−0.9966(.089)	−0.9971(.088)[.087]	−0.9976(.088)[.086]	−1.0068(.088)	−1.0067(.087)[.087]	−1.0066(.087)[.086]
$\beta_{20} = 2$	2.0055(.097)	2.0036(.097)[.089]	2.0038(.097)[.091]	1.9987(.096)	2.0002(.096)[.086]	2.0007(.096)[.091]
$\alpha_0 = -2$	−1.9920(.061)	−1.9983(.060)[.063]	−1.9991(.061)[.062]	−1.9900(.071)	−1.9957(.071)[.072]	−1.9975(.072)[.071]
$\tau_0 = 1$	1.2271(.257)	1.0662(.217)[.228]	1.0678(.226)[.226]	1.1259(.169)	1.0305(.154)[.157]	1.0311(.158)[.155]
$\sigma_0^2 = 1$	0.5954(.092)	0.9421(.146)[.147]	–	0.5947(.095)	0.9465(.151)[.146]	–
			$n = 50$, $T = 7$			
$\beta_{10} = -1$	−0.9987(.051)	−0.9987(.051)[.050]	−0.9986(.051)[.050]	−1.0007(.048)	−1.0000(.048)[.047]	−1.0000(.048)[.046]
$\beta_{20} = 2$	2.0017(.049)	2.0005(.049)[.045]	2.0006(.049)[.048]	2.0013(.051)	2.0002(.051)[.045]	2.0002(.051)[.048]
$\alpha_0 = -2$	−1.9944(.034)	−2.0003(.034)[.034]	−1.9998(.034)[.035]	−1.9912(.038)	−1.9982(.038)[.038]	−1.9983(.038)[.038]
$\tau_0 = 1$	1.1104(.116)	1.0178(.110)[.112]	1.0171(.109)[.111]	1.0630(.081)	1.0127(.079)[.080]	1.0125(.079)[.080]
$\sigma_0^2 = 1$	0.8084(.076)	0.9783(.092)[.093]	–	0.8122(.076)	0.9792(.091)[.092]	–
			$n = 100$, $T = 3$			
$\beta_{10} = -1$	−1.0028(.061)	−1.0019(.060)[.061]	−1.0021(.060)[.060]	−1.0014(.062)	−1.0012(.062)[.060]	−1.0011(.062)[.060]
$\beta_{20} = 2$	1.9973(.073)	1.9984(.073)[.063]	1.9984(.073)[.068]	1.9995(.061)	2.0001(.061)[.057]	2.0001(.061)[.059]
$\alpha_0 = -2$	−1.9961(.041)	−1.9987(.041)[.040]	−1.9987(.041)[.040]	−1.9933(.054)	−1.9984(.054)[.054]	−1.9986(.054)[.054]
$\tau_0 = 1$	1.1842(.185)	1.0329(.151)[.153]	1.0340(.153)[.154]	1.0781(.106)	1.0205(.100)[.103]	1.0176(.100)[.101]
$\sigma_0^2 = 1$	0.6109(.072)	0.9705(.114)[.111]	–	0.6193(.069)	0.9644(.107)[.108]	–
			$n = 100$, $T = 7$			
$\beta_{10} = -1$	−0.9993(.036)	−0.9995(.035)[.035]	−0.9995(.035)[.035]	−1.0001(.030)	−1.0005(.030)[.032]	−1.0005(.030)[.031]
$\beta_{20} = 2$	1.9982(.038)	1.9986(.038)[.034]	1.9986(.038)[.037]	1.9990(.037)	1.9986(.037)[.034]	1.9987(.037)[.037]
$\alpha_0 = -2$	−1.9955(.023)	−1.9987(.023)[.023]	−1.9987(.023)[.023]	−1.9958(.029)	−2.0001(.029)[.029]	−1.9997(.029)[.029]
$\tau_0 = 1$	1.0540(.074)	1.0089(.071)[.072]	1.0090(.072)[.072]	1.0453(.059)	1.0046(.057)[.058]	1.0048(.057)[.057]
$\sigma_0^2 = 1$	0.8362(.053)	0.9947(.063)[.066]	–	0.8258(.057)	0.9892(.069)[.068]	–

Notes: We report the empirical mean (standard deviation) [average asymptotic standard error].

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